Modern Dynamic Asset Pricing Models

Teaching Notes 6.

Consumption, Portfolio Allocation and Equilibrium with Constraints [1]

Pietro Veronesi

University of Chicago
CEPR, NBER

[1]These teaching notes draw heavily on Cuoco (1997) and Karatzas and Shevre (1999, Chapter 6). They are intended for students of Business 35907 only. Please, do not distribute without my prior consent.

Introduction

- Solving a portfolio problem with constraints is difficult and in this teaching notes my intent is to give only a broad understanding of the new techniques that have recently been introduced.
- The articles that progressed our understanding of these problems are listed in the syllabus. In this teaching notes we will follow the approach of Cuoco (1997), who also gives a lucid review of the main points on the previous literature.

The (Usual) Setup

- As before, fix a complete probability space $(\Omega, \mathcal{F}, P)$ on which a n-dimensional Brownian motion $\mathbf{B}$ is defined along with its (augmented) filtration $\{\mathcal{F}_t\}$.
- The consumption space L_+ is the set of non-negative, adapted and integrable consumption processes.
- Let there be n risky assets and a riskless asset

$$\mathbf{S}_t = \mathbf{S}_0 + \int_0^t \mathbf{I}_S \boldsymbol{\mu}_u du + \int_0^t \mathbf{I}_S \boldsymbol{\sigma}_u d\mathbf{B}_u$$

$$\beta_t = e^{\int_0^t r_u du}$$

- where again $\mathbf{I}_S$ is the $n \times n$ diagonal matrix with S^i on the ii-th position.
- Assume that the process r_u is uniformly bounded and that the usual regularity (integrability) conditions hold for the asset prices.
- Assume that $\boldsymbol{\sigma}$ is invertible almost everywhere and define the market price of risk vector

$$\boldsymbol{\nu}_t = \boldsymbol{\sigma}_t^{-1} (\boldsymbol{\mu}_t - r_t \mathbf{1}_n)$$

- Notice that $\boldsymbol{\sigma}$ invertible makes markets complete *if there are no constraints*.
- Assume also that the Novikov's condition is satisfied

$$E\left(\exp\left(\frac{1}{2}\int_0^T \boldsymbol{\nu}_t'\boldsymbol{\nu}_t dt\right)\right) < \infty$$

- As in TN1, we assume that investor's preferences are represented by the utility function

$$U(c) = E\left[\int_0^T u(c,t)\,dt\right]$$

- Some restrictions on the utility functions are imposed but I refer you to the original works (see e.g. Cuoco (1997) or Karatzas and Shreve (1998)). Here, Condition A in TN1 should suffice for the results.
- Finally, investors are endowed with an initial wealth $w_0 \geq 0$ and a stochastic endowment process $e \geq 0$ such that for some K_e we have

$$\int_0^T \beta_t^{-1} e_t \leq K_e$$

Portfolio Constraints

- We now assume that investors cannot trade freely as in previous cases, but they are constrained in keeping their strategies in some set A.
- For modeling purposes, it is convenient to define a trading strategy in *dollar terms* rather than *units* as we did so far.
- Given a trading strategy $\left(\theta^0, \boldsymbol{\theta}\right)$ we shall concentrate on the dollar equivalents $\left(\varphi^0, \boldsymbol{\varphi}\right)$ defined as

$$\varphi_t^0 = \theta_t^0 \beta_t \text{ and } \varphi_t^i = \theta_t^i S_t$$

- Clearly, the trading strategy $\left(\varphi^0, \boldsymbol{\varphi}\right) \in \mathcal{R}^{n+1}$.
- Hence, we can now define in general terms a portfolio constraint set $A \subseteq \mathcal{R}^{n+1}$ such that the dollar positions $\left(\varphi^0, \boldsymbol{\varphi}\right)$ are constrained to lie in A rather than the whole $\mathcal{R}^{n+1}$.
- We shall assume that A is non-empty, closed and convex.
- Examples:

1. No Constraints: $A = \mathcal{R}^{n+1}$.
2. Non-tradeable assets (incomplete markets):

$$A = \left\{ \left(\varphi^0, \boldsymbol{\varphi}\right) \in \mathcal{R}^{n+1} : \varphi^k = 0 \text{ for } k = m+1, ..., n \right\}$$

3. Short-Sale Constraints:

$$A = \left\{ \left(\varphi^0, \boldsymbol{\varphi} \right) \in \mathcal{R}^{n+1} : \varphi^k \geq 0 \text{ for } k = m+1, ..., n \right\}$$

4. Buying Constraints:

$$A = \left\{ \left(\varphi^0, \boldsymbol{\varphi} \right) \in \mathcal{R}^{n+1} : \varphi^k \leq 0 \text{ for } k = m+1, ..., n \right\}$$

5. Minimum Capital Requirements:

$$A = \left\{ \left(\varphi^0, \boldsymbol{\varphi} \right) \in \mathcal{R}^{n+1} : \varphi^0 + \sum_{k=1}^{n} \varphi^k \geq K \right\}$$

for $K \geq 0$. A special case is the portfolio insurance program, where $K > 0$.

- Other constraints can also be introduced (see Cuoco (1997)).

- Given a constraint set A, a consumption plan is said to be $A-$*feasible* if there exists a trading strategy $\left(\varphi^0, \boldsymbol{\varphi}\right)$ such that

$$\begin{aligned}
\varphi_t^0 + \boldsymbol{\varphi}_t &= w_0 + \int_0^t \varphi_u^0 r_u + \boldsymbol{\varphi}_u \mu_u du + \int_0^t \boldsymbol{\varphi}_u \boldsymbol{\sigma}_u d\mathbf{B}_u \\
&\quad - \int_0^t \left(c_u - e_u\right) du \\
\varphi_t^0 + \boldsymbol{\varphi}_t &\geq -K \\
\varphi_T^0 + \boldsymbol{\varphi}_T &= 0
\end{aligned}$$

- and $\left(\varphi_t^0, \boldsymbol{\varphi}_t\right) \in A$ for all $t \in [0, T]$.
- Before turning to the solution of these problems, recall the approach in complete markets.
- We have the following ingredients

  1. A Martingale

$$\xi_t = \exp\left(\int_0^t -\boldsymbol{\nu}_u' d\mathbf{B}_u - \int_0^t \boldsymbol{\nu}_u' \boldsymbol{\nu}_u du\right)$$

  2. A (uniquely defined) state-price density

$$\pi_t = \beta_t^{-1} \xi_t$$

3. A static budget constraint

$$E\left[\int_0^T \pi_t\left(c_t - e_t\right) dt\right] \leq w_0$$

- We then find a saddle point of the Lagrangian

$$L\left(c, \lambda\right) = E\left[\int_0^T u\left(c_t, t\right) dt\right] - \lambda E\left[\int_0^T \pi_t\left(c_t - e_t\right) dt - w_0\right]$$

- The problem is that while with complete markets and no constraints, there is a unique state price density π_t that is consistent with no-arbitrage (see TN1), when there are constrains there are infinitely many state-price densities that are consistent with no-arbitrage.
- As a consequence, we no longer have only *one* static budget constraint, but a whole family of budget constraints that must be satisfied to ensure that the portfolio policies lie within the constraint set A.
- This clearly makes the problem more difficult to solve.
- In order to state more generally the problem, we must introduce some more notation.

Support Functions

- Let the constraint set $A \subseteq \mathcal{R}^{n+1}$ be given.
- For any vector $\mathbf{v} = \left(v^0, \mathbf{v}^1\right) \in \mathcal{R} \times \mathcal{R}^n$, let

$$\delta(\mathbf{v}) = \sup_{\left(\varphi^0, \boldsymbol{\varphi}\right) \in A} -\left(\varphi^0 v^0 + \boldsymbol{\varphi} \cdot \mathbf{v}^1\right)$$

- This is called *support function* of $-A$.
- This function can easily reach $+\infty$ and hence it is important to define its *effective domain*

$$\widetilde{A} = \left\{\mathbf{v} \in \mathcal{R}^{n+1} : \delta(\mathbf{v}) < \infty\right\}$$

- Some restrictions on δ and $\widetilde{A}$ are generally imposed.
- Notice in particular that $\widetilde{A}$ is a closed, convex cone.
- The cases discussed above typically satisfy these conditions.
- In those examples, we have:
  1. No Constraints: $\widetilde{A} = \{0\}$ and $\delta(\mathbf{v}) = 0$ for all $\mathbf{v} \in \widetilde{A}$

2. Non-tradeable assets (incomplete markets):

$$\begin{aligned} \widetilde{A} &= \left\{ \mathbf{v} \in \mathcal{R}^{n+1} : v^k = 0 \text{ for } k = 0, ..., m \right\} \\ \delta(\mathbf{v}) &= 0 \text{ for } \mathbf{v} \in \widetilde{A} \end{aligned}$$

3. Short-Sale Constraints:

$$\begin{aligned} A &= \left\{ \begin{array}{c} \mathbf{v} \in \mathcal{R}^{n+1} : v^k = 0 \text{ for } k = 0, ..., m \\ v^k \geq 0 \text{ for } k = m+1, ..., n \end{array} \right\} \\ \delta(\mathbf{v}) &= 0 \text{ for } \mathbf{v} \in \widetilde{A} \end{aligned}$$

4. Buying Constraints:

$$\begin{aligned} A &= \left\{ \begin{array}{c} \mathbf{v} \in \mathcal{R}^{n+1} : v^k = 0 \text{ for } k = 0, ..., m \\ v^k \leq 0 \text{ for } k = m+1, ..., n \end{array} \right\} \\ \delta(\mathbf{v}) &= 0 \text{ for } \mathbf{v} \in \widetilde{A} \end{aligned}$$

5. Minimum Capital Requirements:

$$\begin{aligned} \widetilde{A} &= \{ k\mathbf{1}_n : k \geq 0 \} \\ \delta(\mathbf{v}) &= -Kv_0 \text{ for } \mathbf{v} \in \widetilde{A} \end{aligned}$$

A Family of Unconstrained Markets

- Given a constrain set A, one can consider the set $\mathcal{N}$ of $\widetilde{A}$-valued processes $\mathbf{v} = \left(v^0, \mathbf{v}^1\right)$ satisfying

$$E\left[\int_0^T |\mathbf{v}_t'\mathbf{v}_t| dt\right] < \infty$$

- Let $\mathbf{v}$ be given and define now the following processes

$$\begin{aligned}
\beta_t^v &= \exp\left(\int_0^t r_u + v_u^0 du\right) \\
\boldsymbol{\nu}_t^v &= \boldsymbol{\sigma}_t^{-1}\left(\boldsymbol{\mu}_t + \mathbf{v}_t^1 - \left(r_t + v_t^0\right)\mathbf{1}_n\right) \\
\xi_t^v &= \exp\left(\int_0^t -\boldsymbol{\nu}_u^{v\prime} d\mathbf{B}_u - \frac{1}{2}\int_0^t \boldsymbol{\nu}_u^{v\prime}\boldsymbol{\nu}_u^v du\right) \\
\pi_t^v &= \left(\beta_t^v\right)^{-1}\xi_t^v
\end{aligned}$$

- These processes are well defined, and ξ_t^v is a local martingale.
- Consider the restricted class $\mathcal{N}^*$ such that ξ_t^v is in fact a martingale.
- For each process $\mathbf{v} \in \mathcal{N}^*$, we can interpret all these processes as representing an economy under one particular probability measure Q^v which is equivalent to P.

- Consider the discounted price process

$$\widehat{\mathbf{S}}_t^v = (\beta_t^v)^{-1}\,\mathbf{S}_t$$

- By Ito's Lemma, we have

$$d\widehat{\mathbf{S}}_t^v = -\left(r_t + v_t^0\right)\widehat{\mathbf{S}}_t^v dt + (\beta_t^v)^{-1}\,\mu_t \mathbf{I}_S dt + (\beta_t^v)^{-1}\,\mathbf{I}_S \boldsymbol{\sigma}_t d\mathbf{B}_t$$

- By Girsanov's Theorem, we know we can define a Brownian motion under Q^v by

$$d\widehat{\mathbf{B}}_t = d\mathbf{B}_t + \boldsymbol{\nu}_t^v dt$$

- Hence, by substituting back, we have

$$\begin{aligned} d\widehat{\mathbf{S}}_t^v &= -\left(r_t + v_t^0\right)\widehat{\mathbf{S}}_t^v dt + (\beta_t^v)^{-1}\,\boldsymbol{\mu}_t \mathbf{I}_S dt \\ &\quad - (\beta_t^v)^{-1}\,\mathbf{I}_S \boldsymbol{\sigma}_t \boldsymbol{\nu}_t^v + (\beta_t^v)^{-1}\,\mathbf{I}_S \boldsymbol{\sigma}_t d\widehat{\mathbf{B}}_t \\ &= -\left(r_t + v_t^0\right)\widehat{\mathbf{S}}_t^v dt + (\beta_t^v)^{-1}\,\boldsymbol{\mu}_t \mathbf{I}_S dt \\ &\quad - (\beta_t^v)^{-1}\,\mathbf{I}_S\left(\boldsymbol{\mu}_t + \mathbf{v}_t^1 - \left(r_t + v_t^0\right)\mathbf{1}_n\right) \\ &\quad + (\beta_t^v)^{-1}\,\mathbf{I}_S \boldsymbol{\sigma}_t d\widehat{\mathbf{B}}_t \\ &= -(\beta_t^v)^{-1}\,\mathbf{I}_S \mathbf{v}_t^1 dt + (\beta_t^v)^{-1}\,\mathbf{I}_S \boldsymbol{\sigma}_t d\widehat{\mathbf{B}}_t \end{aligned}$$

- Notice an important difference compared to the case of no constraints.
- Now, the discounted price process is no longer (necessarily) a martingale.
- However, given the characterization of the processes $\mathbf{v}^1$, that by definition must lie in $\widetilde{A}$, we can still say something about the properties of the discounted price processes.
- Let us denote by $\widehat{\boldsymbol{\mu}}_t^v$ the drift of the discounted processes. We then have a relationship between the constrain sets A, and the class of equivalent martingale measures Q^v that characterize the stock price process.
- These properties will become useful to solve the consumption problem.

1. No Constraints: $\widetilde{A} = \{0\}$ and hence $\widehat{\boldsymbol{\mu}}_t^v = 0$. Hence, the discounted price processes are (local) martingales.
2. Non-tradeable assets (incomplete markets):
$$\widetilde{A} = \left\{ \mathbf{v} \in \mathcal{R}^{n+1} : v^k = 0 \text{ for } k = 0, ..., m \right\}$$
Hence $\widehat{\mu}_{k,t}^v = 0$ for $k = 1, .., m$. In this case the set $\{Q^v : \mathbf{v} \in \mathcal{N}^*\}$ correspond to the set of probability measures equivalent to P under which the tradeable assets have stock prices that are indeed local martingale.
3. Short-Sale Constraints:
$$\widetilde{A} = \left\{ \begin{array}{c} \mathbf{v} \in \mathcal{R}^{n+1} : v^k = 0 \text{ for } k = 0, ..., m \\ v^k \geq 0 \text{ for } k = m+1, ..., n \end{array} \right\}$$
Hence $\widehat{\mu}_{k,t}^v = 0$ for $k = 1, .., m$ and $\widehat{\mu}_{k,t}^v \leq 0$ for $k = m+1, ..., n$. In this case the set $\{Q^v : \mathbf{v} \in \mathcal{N}^*\}$ corresponds to the set of Q^v under which the unconstrained discounted

stock prices are local martingales and the constrained discounted stock prices are local super martingales.

4. Buying Constraints:

$$\widetilde{A} = \left\{ \begin{array}{c} \mathbf{v} \in \mathcal{R}^{n+1} : v^k = 0 \text{ for } k = 0, ..., m \\ v^k \leq 0 \text{ for } k = m+1, ..., n \end{array} \right\}$$

   Hence $\widehat{\mu}_{k,t}^{v} = 0$ for $k = 1, .., m$ and $\widehat{\mu}_{k,t}^{v} \geq 0$ for $k = m+1, ..., n$. Hence, the unconstrained discounted assets are local martingales and the constrained discounted assets are local sub-martingales.

5. Minimum Capital Requirements:

$$\widetilde{A} = \{k\mathbf{1}_n : k \geq 0\}$$

   Hence, $\widehat{\mu}_{k,t}^{v} \leq 0$ so that the discounted stock price process are local supermartingale, with drift proportional to the stock price.

- Another way of interpreting each $\mathbf{v} \in \mathcal{N}^*$ is that it generates the unique state price density π_t^v in a *fictituous unconstrained economy* where the parameters characterizing the economy are given by

$$\begin{aligned} r_t^v &= r_t + v_t^0 \\ \boldsymbol{\mu}_t^v &= \boldsymbol{\mu}_t + \mathbf{v}_t^1 \end{aligned}$$

- For each of these problems, we know how to solve the maximization problem.

- In addition, it turns out that at this optimum for the fictituous uncostrained economy the agent is happy to be "constrained," in the sense that the constrain is non-binding in this fictituous economy.

- One way to tackle the constrained maximization problem is to solve the fictitious unconstrained maximization problem with the different state-price density and then maximize the over the set of utilities so obtained.

- The following result shows that indeed, each state price density π_t^v with $\mathbf{v} \in \mathcal{N}^*$ constitutes an arbitrage free state-price density in the *original* economy with constraints and that the satisfaction of a static budget constraint with respect to all these state price densities is also sufficient to guarantee $A-$feasibility.

- **Proposition 1**: A consumption process c is $A-$feasible if and only if for all $\mathbf{v} \in \mathcal{N}^*$

$$E^{Q^v}\left[\int_0^T (\beta_t^v)^{-1}(c_t - e_t)\,dt\right] \le w_0 + E^{Q^v}\left[\int_0^T (\beta_t^v)^{-1}\delta(\mathbf{v}_t)\,dt\right]$$

- Proof: See e.g. Cuoco (1997).

- The method used to prove this theorem is similar, albeit more elaborate, to the one used to show a similar proposition in TN1. It is good to go through it once in life, but we won't do it here.

- Hence, we can now reformulate the maximization problem of the investor as

$$\max_{c \in \mathcal{C}} U(c) \tag{1}$$

- subject to

$$E^{Q^v}\left[\int_0^T (\beta_t^v)^{-1}(c_t - e_t - \delta(\mathbf{v}_t))\,dt\right] \leq w_0 \text{ for all } \mathbf{v} \in \mathcal{N}^* \tag{2}$$

- and where $\mathcal{C}$ is a set of feasible consumption plans (see restrictions in Cuoco (1997)).
- Notice that for most cases, we have that $\delta(\mathbf{v}_t) = 0$, which implies an identical budget constraint as the one we looked at in TN2.
- However, now there is a continuum of budget constraints.
- It turns out that this messes up things *a lot*. Some key properties of the space of integrable functions c ($\mathcal{L}^1$ spaces) to admit a solution to the convex problem (1) are missing.

Convex Duality Approach (Cvitanic and Karatzas (1992))

- Cvitanic and Karatzas (1992) used a *convex duality* approach to solve the problem (1) and were able to prove existence of a consumption plan under somewhat restrictive assumptions on the utility functions and on the endowment process e.
- Given its applications in the literature, we review here the main points. We still follow Cuoco (1997) main arguments.
- Let c^* be the optimal consumption plan and assume that there exists a $\mathbf{v}^* \in \mathcal{N}^*$, such that

$$E^{Q^{v^*}}\left[\int_0^T \left(\beta_t^{v^*}\right)^{-1}\left(c_t^* - e_t - \delta\left(\mathbf{v}_t^*\right)\right) dt\right] = w_0 \tag{3}$$

- Since the set $\{\pi^v : \mathbf{v} \in \mathcal{N}^*\}$ is a convex set, there should be a state price density π^{v^*} and a Lagrangian multiplier λ^* such that $(c^*, \lambda^*, \mathbf{v}^*)$ is a *saddle point* of the function

$$\mathcal{L}\left(c, \lambda, \mathbf{v}\right) = U\left(c\right) - \lambda E\left[\int_0^T \pi_t^v\left(c_t - e_t - \delta\left(\mathbf{v}_t\right)\right) dt - w_0\right] \tag{4}$$

- where one maximizes over c and minimizes over $(\lambda, \mathbf{v})$.

- Notice that we can rewrite

$$\begin{aligned}\mathcal{L}(c,\lambda,\mathbf{v}) &= E\left[\int_0^T u(c_t,t)-\lambda\left(\pi_t^v\left(c_t-e_t-\delta(\mathbf{v}_t)\right)dt-w_0\right)\right] \\ &= E\left[\int_0^T \left(u(c_t,t)-\lambda\pi_t^v c_t\right)dt\lambda\left(w_0+\int_0^T \pi_t^v\left(e_t+\delta(\mathbf{v}_t)\right)dt\right)\right]\end{aligned}$$

- Notice that "c" only enters in the term $\left(u(c_t,t)-\lambda\pi_t^v c_t\right)$.
- Hence, we can define the function

$$\widetilde{u}(z,t)=\sup_{c\geq 0}\left[u(c,t)-zc\right] \tag{5}$$

- This function $\widetilde{u}(z,t)$ is called *convex conjugate* of $-u(-c,t)$.
- If the Inada Conditions are satisfied, the problem (5) is solved by

$$c=\mathcal{I}_u(z,t)$$

- where $\mathcal{I}_u(.,t)$ is the inverse of the marginal utility $u_c(c,t)$.
- If we first maximize over c, we can now define the minimization program

$$\min_{\lambda,\mathbf{v}}\mathcal{J}(\lambda,\mathbf{v})=E\left[\int_0^T \widetilde{u}\left(\lambda\pi_t^v,t\right)dt+\lambda\left(w_0+\int_0^T \pi_t^v\left(e_t+\delta(\mathbf{v}_t)\right)dt\right)\right] \tag{6}$$

- The program (6) is called *Dual (Shadow State-Price) Problem* of the original problem (1).
- The key result is the following (see Cuoco (1997))
- **Proposition 1**. Suppose that the utility function $u(c,t)$ satisfies the Inada Conditions and there exists a constant $\beta \in (0,1)$ and $\gamma \in (0,\infty)$ such that for all $(c,t) \in (0,\infty) \times [0,T]$

$$\beta u_c(c,t) \geq u_c(\gamma c,t) \tag{7}$$

- If these exists a solution $(\lambda^*, \mathbf{v}^*)$ to the dual problem (6), and

$$E\left[\int_0^T \pi_t^{v^*} \left(\mathcal{I}_u\left(\lambda^* \pi_t^{v^*}, t\right) - e_t - \delta\left(\mathbf{v}_t^*\right)\right) dt\right] < \infty$$

- then there exists a constrained optimal consumption plan c^* such that

$$u_c(c_t^*, t) = \lambda^* \pi_t^{v^*} \tag{8}$$

- holds and

$$E\left[\int_0^T \pi_t^{v^*} \left(\mathcal{I}_u\left(\lambda^* \pi_t^{v^*}, t\right) - e_t - \delta\left(\mathbf{v}_t^*\right)\right) dt\right] = w_0 \tag{9}$$

- Conversely, if (8) and (9) hold for some $(\lambda^*, \mathbf{v}^*) \in (0,\infty) \times \mathcal{N}^*$ and some A-feasible consumption plan c^*, then $(\lambda^*, \mathbf{v}^*)$ solves the dual problem.

- This proposition gives us a tool to solve some interesting situations. We will use it several times.
- Notice that condition (7) is satisfied by all the CRRA (Constant Relative Risk Aversion) utility functions

$$u\left(c_t, t\right) = \frac{e^{-\phi t} c_t^{1-b}}{1-b}$$

- with $b \in (0, \infty]$.
- In fact,

$$u_c\left(\gamma c_t, t\right) = e^{-\phi t}\left(\gamma c\right)^{-b} = \gamma^{-b} e^{-\phi t} c^{-b} = \frac{1}{\gamma^b} u_c\left(c, t\right)$$

- Hence. we can find $\beta \in (0, 1)$ and $\gamma \in (0, \infty)$ such that the condition is satisfied.

Example: Log Utility

- To illustrate the above findings within the context of an example, consider the optimal consumption and investment strategy of a log-utility investor in the presence of short-sale constraints or incomplete markets.
- Suppose that the constraint applies to all the securities (for simplicity). In the case of "incomplete markets" it means that the investors cannot invest in the stock market (we shall use the results obtained here in TN4).
- Finally, assume no stochastic income: $e_t = 0$.
- From above, we have that the effective domain and the support functions are

$$\begin{aligned} \widetilde{A} &= \left\{ \mathbf{v} \in \mathcal{R}^{n+1} : v^0 = 0 \text{ and } \mathbf{v}^1 \geq 0 \right\} \\ \delta(\mathbf{v}) &= 0 \end{aligned}$$

- for short sale constraints and

$$\begin{aligned} \widetilde{A} &= \left\{ \mathbf{v} \in \mathcal{R}^{n+1} : v^0 = 0 \right\} \\ \delta(\mathbf{v}) &= 0 \end{aligned}$$

- for incomplete markets.

- In addition, we also have that $\mathcal{I}_u(z,t) = e^{-\phi t} z^{-1}$ which implies

$$\begin{aligned}\widetilde{u}(z,t) &= u(\mathcal{I}_u(z,t),t) - z\mathcal{I}_u(z,t)\\ &= e^{-\phi t}\log\left(z^{-1}e^{-\phi t}\right) - e^{-\phi t}\\ &= -e^{-\phi t}(1+\phi t+\log(z))\end{aligned}$$

- Hence, we obtain that the $\mathcal{J}(\lambda,\mathbf{v})$ in (6) is given by

$$\begin{aligned}\mathcal{J}(\lambda,\mathbf{v}) &= E\left[\int_0^T -e^{-\phi t}\left(1+\phi t+\log\left(\lambda\pi_t^v\right)\right)dt + \lambda w_0\right]\\ &= -\int_0^T e^{-\phi t}(1+\log(\lambda))\,dt - \int_0^T e^{-\phi t}\phi t dt + \lambda w_0\\ &\quad -E\left[\int_0^T e^{-\phi t}\log\left(\pi_t^v\right)dt\right]\\ &= (1+\log(\lambda))\frac{e^{-\phi T}-1}{\phi} + \frac{e^{-\phi T}\phi T + e^{-\phi T}-1}{\phi} + \lambda w_0\\ &\quad +E\left[\int_0^T e^{-\phi t}\left(\int_0^t \boldsymbol{\nu}_u^{v\prime}d\mathbf{B}_u + \int_0^t\left(r_u+v_u^0+\frac{1}{2}\boldsymbol{\nu}_u^{v\prime}\boldsymbol{\nu}_u^v\right)du\right)dt\right]\\ &= (2+\log(\lambda))\frac{e^{-\phi T}-1}{\phi} + e^{-\phi T}T + \lambda w_0\\ &\quad +\int_0^T e^{-\phi t}E\left(\int_0^t\left(r_u+v_u^0+\frac{1}{2}\boldsymbol{\nu}_u^{v\prime}\boldsymbol{\nu}_u^v\right)du\right)dt\end{aligned}$$

- where we used the fact that

$$\begin{aligned}\pi_t^v &= (\beta_t^v)^{-1}\,\xi_t^v \\ &= \exp\left(\int_0^t -\left(r_u + v_u^0 + \frac{1}{2}\boldsymbol{\nu}_u^{v\prime}\boldsymbol{\nu}_u^v\right)du - \boldsymbol{\nu}_u^{v\prime} d\mathbf{B}_u\right)\end{aligned}$$

- Hence, the $\mathcal{J}(\lambda, \mathbf{v})$ is additively separable in the two arguments and we can minimize one at the time.
- With respect to λ, we find that $\mathcal{J}_\lambda(\lambda, \mathbf{v}) = 0$ implies

$$\frac{1}{\lambda}\frac{e^{-\phi T} - 1}{\phi} + w_0 = 0$$

- which yields

$$\lambda^* = \frac{1}{w_0}\frac{1 - e^{-\phi T}}{\phi}$$

- We now have to minimize the second term

$$\int_0^T e^{-\phi t} E\left(\int_0^t \left(r_u + v_u^0 + \frac{1}{2}\boldsymbol{\nu}_u^{v\prime}\boldsymbol{\nu}_u^v\right)du\right)dt$$

- with respect to $\mathbf{v} \in \widetilde{A}$. Recall first that we must have $v_u^0 = 0$, by definition of $\widetilde{A}$.

- The minimum is obtained by *pointwise* minimizing the expression

$$\begin{aligned}\boldsymbol{\nu}_t^{v\prime}\boldsymbol{\nu}_t^v &= \left(\boldsymbol{\mu}_t + \mathbf{v}_t^1 - r_t\mathbf{1}_n\right)'\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\right)^{-1}\left(\boldsymbol{\mu}_t + \mathbf{v}_t^1 - r_t\mathbf{1}_n\right)\\ &= \boldsymbol{\nu}_t'\boldsymbol{\nu}_t + \mathbf{v}_t^{1\prime}\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\right)^{-1}\mathbf{v}_t^1 + 2\mathbf{v}_t^1\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\right)^{-1}\left(\boldsymbol{\mu}_t - r_t\mathbf{1}_n\right)\end{aligned}$$

- Recall that $\mathbf{v}_t^1$ are constrained to lie in $\widetilde{A}$, though.
- The short-sale constraint implies that $\mathbf{v}_t^1 \geq 0$, which if $\left(\boldsymbol{\mu}_t - r_t\mathbf{1}_n\right) > 0$ implies that $\mathbf{v}_t = 0$ obtains the minimum.
- To be more specific, suppose there is **only one risky asset**. Then, the unconstrained minimization would be

$$v_t^1 = -\left(\mu_t - r_t\right)$$

- Hence, the constrained optimum is

$$v_t^{1*} = \max\left(-\left(\mu_t - r_t\right), 0\right)$$

- This is intuitive: If $\mu_t - r_t > 0$, a log utility investor (who is myopic) will invest a positive amount in the stock, hence the short-sale constraint is not binding and $v_t^{1*} = 0$. In this case, since also $v_t^0 = 0$, we have that the state-price density is the same as in the unconstrained case

$$\pi_t^{v^*} = \beta_t^{-1}\xi_t \text{ (if } \mu_t - r_t > 0)$$

- If instead $(\mu_t - r_t) < 0$, it would be profitable for a myopic investor to short the risky asset, but here the constraint kicks in and the fictituous market price of risk becomes

$$\nu_t^{v^*} = \sigma_t^{-1}\left(\mu_t + v_t^{*1} - r_t\right) = 0$$

- Hence, the fictituous state price density becomes

$$\pi_t^{v^*} = \beta_t^{-1}\xi_t^{v^*} = \beta_t^{-1}$$

- That is, it is as if there are no other assets in the fictituous economy!
- And for our investor this is exactly the case because of the short-sale constraint.
- A similar reasoning goes for the investor always barred to invest in the stock market.
- In this case we have that $\mathbf{v}_t^1$ is unconstrained (while $v_t^0 = 0$) which implies that it must be given by

$$\mathbf{v}_t^1 = -\left(\boldsymbol{\mu}_t - r_t\mathbf{1}_n\right)$$

- Substituting this into the state price density, we obtain (again) that

$$\pi_t^{v^*} = \beta_t^{-1}\xi_t^{v^*} = \beta_t^{-1}$$

- The fictituous economy is made up only by the risk-free asset and the investor optimally chooses there only.

- In summary, we see that rather than solving the constrained maximization problem directly, the approach taken here is to change the state-price density $\pi_t^{v^*}$ of the investor in such a way that an unconstrained agent facing this particular state-price density would never hit the constraints and, in addition, his/her optimal choices are identical to the one that the constrained agent would choose.

- As a final step, given the results above about the optimal $\mathbf{v}^*$ and λ^*, we can solve for the optimal consumption and asset allocation.

- That is, we have just to use the usual results about optimal allocation that we derived in previous teaching notes, but use the state-prices π_t^{v*}.

- The optimal consumption is

$$c_t^* = e^{-\phi t} \frac{1}{\lambda^* \pi_t^{v^*}} = e^{-\phi t} \frac{w_0 \phi}{\left(1 - e^{-\phi T}\right) \pi_t^{v^*}}$$

- The process for wealth is

$$\begin{aligned} W_t &= \frac{1}{\pi_t^{v^*}} E\left[\int_t^T \pi_\tau^{v^*} c_\tau^* d\tau\right] \\ &= \frac{1}{\pi_t^{v^*}} E\left[\int_t^T \pi_\tau^{v^*} e^{-\phi\tau} \frac{w_0 \phi}{\left(1-e^{-\phi T}\right) \pi_\tau^{v^*}} d\tau\right] \\ &= \frac{1}{\pi_t^{v^*}} \frac{w_0 \phi}{\left(1-e^{-\phi T}\right)} \frac{e^{-\phi t}-e^{-\phi T}}{\phi} \\ &= \frac{1}{\pi_t^{v^*}} \frac{w_0}{\left(1-e^{-\phi T}\right)}\left(e^{-\phi t}-e^{-\phi T}\right) \end{aligned}$$

- Notice that we can rewrite the consumption process as

$$c_t^* = \frac{\phi}{\left(1-e^{-\phi(T-t)}\right)} W_t$$

- Finally, the optimal trading strategy is

$$\varphi_t = \max\left(0, -\left(\mu_t - r_t\right) W_t\right)$$

- if there are short-sale constraints, or

$$\varphi_t = 0$$

- if (clearly) the agent is forbidden from investing in the market.

Direct Approach (Cuoco (1997))

- The conditions to prove existence in the Dual problem (6) are very restrictive.
- One needs to have $\delta = 0$, $e = 0$ and a relative risk aversion always below 1.
- Cuoco (1997) tackled directly the problem (1) by using a technique called of "relaxation projection," and could prove existence of a solution under much more general conditions (see Theorem 2).
- In particular, all the utility functions of the HARA (Hyperbolic Absolute Risk Aversion) class satisfy these conditions

$$u(c,t) = \frac{be^{\phi t}}{1-b}\left(\frac{\alpha c_t}{b} + \beta\right)^{1-b}$$

- with $b \in (0,\infty], \alpha > 0$ and $\beta, \phi \geq 0$ with $\beta = 1$ if $b = \infty$.
- Rather than going over the existence result, we pass directly to characterize the optimal consumption policies and obtain implications for stock returns.
- In the unconstrained case we could make use of the Lagrange multiplier to show that the optimal consumption equal the inverse of the utility function defined at a point proportional to the current state price density.

- That is, we recall that in the unconstrained case we obtained (in TN3) that

$$c_t^* = \mathcal{I}_u \left(\lambda \pi_t, t \right)$$

- where π_t is uniquely defined.
- The next proposition generalizes this result: (Cuoco 1997)
- **Proposition 2**: Let c_t^* be the optimal consumption plan and assume that $c_t^* \neq 0$. Suppose that there exists a λ such that

$$E\left[\int_0^T u_c\left(\lambda c_t^*, t\right) c_t dt\right] < \infty \tag{10}$$

- Then there exists a sequence $(\psi_n \pi^{v_n})$ with $\psi_n > 0$ and $\mathbf{v}_n \in \mathcal{N}^*$ such that

$$\left(u_c\left(c_t^*\left(\omega\right), t\right) - \psi_n \pi_t^{v_n}\left(\omega\right)\right) c_t^*\left(\omega\right) \longrightarrow 0 \tag{11}$$

- almost everywhere
- If in addition

$$\inf_{\mathbf{v}_n \in \mathcal{N}^*} E\left[\int_0^T \pi_t^v c_t^*\right] < \infty \tag{12}$$

- then (11) holds with $\psi_n = \psi$ for all n.

- An implication of this is that if $c^* > 0$ (and this happened under condition A in TN1), we then have
- **Corollary 1**: If $c^* > 0$ almost everywhere, and (10) and (12) hold, then there exists $\psi > 0$ and a sequence of state-price densities $\{\pi^{v_n}\}$ with $\mathbf{v}_n \in \mathcal{N}^*$ such that

$$u_c\left(c_t^*\left(\omega\right),t\right) = \lim_{n \longrightarrow \infty} \psi \pi_t^{v_n}\left(\omega\right)$$

- for almost all $(t, \omega) \in [0, T] \times \Omega$.
- To recapitulate, what this corollary shows is that in the presence of portfolio constraints (and incomplete markets), there are a continuum of state-price densities that are consistent with the notion of no-arbitrage.
- However, there exists a sequence of state price densities converging (state-by-state) to some state price density π_t^{v*} such that the optimal consumption still satisfies the

$$u_c\left(c_t^*, t\right) = \psi \pi_t^{v*}$$

- as it was true for the unconstrained case.

A Constrained C-CAPM Result

- We end these teaching notes with an additional results on the nature of the equilibrium when there are constraints in the economy.
- As in TN2, suppose there are m agents, each endowed with an instantaneous utility function $u^i(c)$ and a common discount rate ϕ.
- We say that the agent is at a regular optimum if the marginal utility process at the optimum consumption plan is proportional to a "generalized state price density" $\Psi\beta^{-1}\xi^v$, where Ψ is a non-increasing process and $\mathbf{v} \in \mathcal{N}$.
- That is, we assume that for all i we have

$$u_c^i\left(c_t^i\right) = e^{\phi t}\Psi_t^i\beta_t^{-1}\xi_t^{v_i}$$

- where $\mathbf{v}_i \in \mathcal{N}$ and Ψ_t^i is a non-increasing process.
- Notice that the necessity to introduce a process Ψ_t^i stems from the fact that in the "fictituous economy" for agent i, we still need to discount future cash flows at the economy-wide interest rate r_t.
- In other words, the state-price density for each agent we analyzed in the previous sections was

$$\pi_t^v = \left(\beta_t^v\right)^{-1}\xi_t^v$$

- where

$$\beta_t^v = \exp\left(\int_0^t r_u + v_u^0 du\right) = \beta_t \exp\left(\int_0^t v_u^0 du\right)$$

- Since across agents we must have the same bond price β_t, we now need to introduce a new process that transforms the constrained optimization problem of each agent into a fictituous unconstrained problem.

- Let the aggregate endowment be denoted as

$$e_t = \sum_{i=1}^m e_t^i = \sum_{i=1}^m \widehat{c}_t^i$$

- where $\widehat{c}^i$ is the optimal consumption of agent i.

- We then have the following interesting result (by Cuoco (1997))

- **Proposition 3**: Under regularity conditions on the utility function, the equilibrium risk premia are determined by

$$E_t\left[\frac{dS_t^i}{S_t^i}\right] - r_t = \Gamma(\mathbf{c}_t) \times Cov\left(\frac{dS_t^i}{S_t^i}, de_t\right) \quad (13)$$

$$-\Gamma(\mathbf{c}_t) \sum_{j=1}^m a_j \left(\widehat{c}_t^j\right)^{-1} \left(v_{j,t}^i - v_{j,t}^0\right) \quad (14)$$

- where I recall from TN5 that

$$a_j\left(\widehat{c}_t^j\right) = -\frac{u_{cc}^j\left(\widehat{c}_t^j\right)}{u_c^j\left(\widehat{c}_t^j\right)}$$

- is the absolute risk aversion coefficient of agent j, and

$$\Gamma\left(\mathbf{c}_t\right) = \frac{1}{\sum_{j=1}^m a^j\left(\widehat{c}_t^j\right)^{-1}}$$

- is the coefficient of absolute risk aversion of the market itself.
- Equation (13) is the constrained version of the C-CAPM.
- The first term is the usual risk premium coefficient that is generated by covariance of the return of asset i with the aggregate endowment.
- The second component in line (14) stems from the constraints imposed on the optimal portfolio and/or the incompleteness of markets.
- In fact, the latter element is a weighted average of the processes $\mathbf{v}_j$ that identify the shadow state-prices for agent j in the economy, when he/she is subject to constraints.
- Notice that we can see immediately from the form of $\widetilde{A}$ what are the implications for the C-CAPM when we impose various constraints that are homogenous across individuals.

- For example, in the case of *incomplete markets*, we recall that we have $v_t^i = 0$ for $i = 0, ..., m$ (where m is the number of traded assets) which implies immediately that for the assets that *are* traded, the C-CAPM holds.
- Similarly, for the case of short-sale constraints, we know that $v_t^i = 0$ for $i = 0, ..., m$ for the m unconstrained assets while $v_t^i \geq 0$ for the constrained asset.
- This implies that for the unconstrained assets the C-CAPM works while for the constrained assets the C-CAPM over-predicts the returns.
- Similar implications can be derived in the other examples.

References

Cuoco (1997) "Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income" Journal of Economic Theory, 72, 33-73.

Cvitanic and Karatzas (1992) Convex Duality in Constrained Portfolio Optimization, Annals of Applied Probability, 2, 767-818.