Teaching Notes #4

Alternative Preferences: Habit Formation and Recursive Utility[1]

Pietro Veronesi
Graduate School of Business
University of Chicago
Business 35909 Spring 2005

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[1]These teaching notes draw heavily on the papers quoted in the references or in the Syllabus. They are intended for students of Business 537 only. Please, do not distribute without my prior consent.

Introduction

- In this teaching notes we are going to examine a few recent models that try to address the various empirical "puzzles" by modifying the preferences of agents.
- The topics we shall cover are the following
  1. External Habit Formation as in Abel (1990) (applied to current set up, and consistent with learning
  2. External Habit Formation as in Campbell and Cochrane (1999) (use the setting of Santos and Veronesi (2005))
  3. Internal Habit Formation as in Detemple and Zapatero (1991)
  4. Stochastic Differential Utility (Recursive Utility) as in Duffie and Epstein (1992).
- Notice that we leave out many, many important papers. The reason is to keep the set up as close as possible to the one introduced in previous teaching notes. See the syllabus for other very important contributions in the field.

1 Habit Formation

- We now discuss the Habit Formation models, still in the context of a standard Lucas economy.
- We consider first a modification to Abel (1990) set up – with external habit – and then we move to an internal habit formation model.

2 External Habit with Abel (1990) preferences

- Consider a standard pure-exchange economy (Lucas (1978)) populated by a continuum of identical investors with utility function
$$u(C_t, X_t) = \frac{(C_t/X_t)^{1-\gamma}}{1-\gamma}.$$
where γ is the coefficient of relative risk aversion, ϕ the discount rate and X_t is the habit level discussed below.
- I assume that total *log* endowment (aggregate consumption) evolves according to the process
$$de_t = \theta_t dt + \sigma dB_t^1$$
- Consumption data suggest a small predictability compoenent in expected consumption growth.[2] Let's assume the drift rate of aggregate endowment is assumed to evolve according to the

[2] Using data from 1947 - 2002, the log-likelihood of a constant drift is 520, while the one with time varying drift is 834. Although a log-likelihood ratio test does not have standard distribution in this case, such a difference in log-likelihoods would most likely reject the null of constant drift at any confidence level.

mean reverting process

$$d\theta_t = k\left(\overline{\theta} - \theta_t\right)dt + \sigma_\theta dB_t^1$$

- Notice perfect markets, as dB_t^1 drives both the endowment e_t and the expected growth rate of endowment θ_t.
  - As shown in TN 3, this could be due to learning about θ
- We assume that the external habit X_t is given by a geometric exponentially weighted average of past endowment realizations, that is

$$x_t = \log X_t = x_0 e^{-\alpha t} + \alpha \int_0^t e^{-\alpha(\tau - t)} e_\tau d\tau$$

- This implies that log habit follows the process

$$\begin{aligned} dx_t &= -\alpha\left(x_0 e^{-\alpha t} + \alpha \int_0^t e^{-\alpha(\tau-t)} e_\tau d\tau\right) dt + \alpha e_t dt \\ &= \alpha\left(e_t - x_t\right) dt \end{aligned}$$

- Let $y_t = e_t - x_t$ which then follows

$$dy_t = \left(\theta_t - \alpha y_t\right) dt + \sigma dB_t^1$$

- Finally, the investment opportunity set is made up of one stock, the market, and one risk free bond.
- The price of stocks is given by

$$dS_t = \mu_S S_t dt + \sigma_S S_t dB_t^1$$

- where μ_t and σ are determined in equilibrium
- The risk free bond yields an instantaneous rate of return r_t.

2.1 The Maximization Problem and the State Price Density

- The maximization problem of each agent is

$$\max_{\left(C,\left(\varphi^0,\varphi\right)\right)} U\left(C,X\right) = E\left[\int_0^\infty e^{-\phi t} u\left(C_t,X_t\right) dt\right]$$

- subject to

$$E\left[\int_0^\infty \pi_t C_t dt\right] \leq E\left[\int_0^\infty \pi_t e_t dt\right]$$

- where

$$\pi_t = e^{\left(\int_0^t -\left(r_u+\frac{1}{2}\nu_u\nu_u'\right)du+\int_0^t \nu_u' dB_u^1\right)}$$

- and

$$\nu_t = \sigma_S^{-1}\left(\mu_S - r_t\right)$$

- As usual, define the Lagrangian

$$\mathcal{L}(C, X, \lambda) = E\left[\int_0^T e^{-\phi t} u(C_t, X_t) - \lambda(\pi_t C_t - \pi_t e_t)\, dt\right] \tag{1}$$

- Here there is nothing different from the set up in TN2, because the agent takes X_t exogenously given.
- Thus, we have that

$$u_C(C_t, X_t) = \lambda \pi_t e^{\phi t}$$

- holds here too. We can renormalize $\lambda = 1$, and keep going.
- In other words, we can write the equilibrium stochastic discount factor as

$$\pi_t = e^{-\phi t - \gamma c_t + (\gamma - 1)x_t} = e^{-\phi t - \gamma y_t - x_t} = e^{-\phi t + (1-\gamma) y_t - c_t}$$

- where, of course, we have $c_t = e_t$.

2.2 The Price of Assets

- We now move to price assets. Consider now an asset that pays the dividend

$$D_t = C_t^{\beta} H_t^{\beta_h}$$

where H_t is a random variable following a stationary stochastic process

$$dh_t = -k_h h_t dt + \boldsymbol{\sigma}_h d\mathbf{B}_t$$

where $h_t = \log(H_t)$.

- Campbell (1986) and Abel (199?) interpret β as a measure of leverage.
- Bansal et al refer to β as consumption leverage. Notice that since $\delta_t = \beta c_t + \beta_h h_t$, and h_t is stationary, the assumption is simply that dividends and consumption are cointegrated series. Bansal et al provide evidence to this effect. Menzly, Santos and Veronesi show that for a number of industry portfolios it is possible to reject no cointegration against the alternative that the cointegrating vector is exactly $(1, -1)$, implying $\beta = 1$.
- Finally, notice that $\beta = 0 = \beta_h$ yields the price of consol bond paying a unit of consumption good every period.

- Moving forward, let $\delta_t = \log(D_t)$ be the log dividend. We have that the dividend of this asset follows the process

$$\begin{aligned} d\delta_t &= \beta dc_t + \beta_h dh_t \\ &= (\beta\theta_t - \beta_h h_t)\, dt + \boldsymbol{\sigma}_\delta d\mathbf{B} \end{aligned}$$

where $\boldsymbol{\sigma}_\delta = \beta_h \boldsymbol{\sigma}_h + \beta \boldsymbol{\sigma}_c$. Thus, β regulates the long-term covariance of D_t and consumption C_t, while the instantaneous covariance is given by

$$Cov(d\delta_t, dc_t) = \beta\sigma_c^2 + \beta_h \boldsymbol{\sigma}_h \boldsymbol{\sigma}_c'$$

- The price of such an asset is given by

$$S_t = E_t\left[\int_t^\infty \frac{\pi_\tau}{\pi_t} D_\tau d\tau\right]$$

$$= \frac{1}{\pi_t} E_t \left[\int_t^\infty e^{-\phi\tau - \gamma y_\tau - x_\tau + \beta c_\tau + \beta_h h_\tau} d\tau \right]$$
$$= \frac{1}{\pi_t} E_t \left[\int_t^\infty e^{-\phi\tau + (1-\gamma) y_\tau + (\beta - 1) c_\tau + \beta_h h_\tau} d\tau \right]$$

- We now make use of the following standard lemma to obtain the value of the asset:

- **Lemma 1**: Define $z_t = (1-\gamma)\, y_t + (\beta - 1)\, c_t + \beta_h h_t$, which then follows the process

$$\begin{aligned} dz_t &= ((\beta-\gamma)\,\theta_t - (1-\gamma)\,\alpha y_t - k_h \beta_h h_t)\, dt \\ &\quad + ((\beta-\gamma)\,\boldsymbol{\sigma}_c + \beta_h \boldsymbol{\sigma}_h)\, d\mathbf{B}_t \end{aligned}$$

Consider the system $\mathbf{N}_t = (z_t, y_t, \theta_t, h_t)$ which follows the linear process

$$d\mathbf{N} = (\mathbf{A}_N + \mathbf{B}_N \mathbf{N})\, dt + \boldsymbol{\Sigma} d\mathbf{B}$$

with

$$A = \begin{pmatrix} 0 \\ 0 \\ k\overline{\theta} \\ 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & -(1-\gamma)\,\alpha & \beta-\gamma & -\beta_h k_h \\ 0 & -\alpha & 1 & 0 \\ 0 & 0 & -k & 0 \\ 0 & 0 & 0 & -k_h \end{pmatrix}$$

$$\text{and } \Sigma = \begin{pmatrix} (\beta-\gamma)\,\boldsymbol{\sigma}_c + \boldsymbol{\sigma}_h \\ \boldsymbol{\sigma}_c \\ \boldsymbol{\sigma}_\theta \\ \boldsymbol{\sigma}_h \end{pmatrix}$$

We then have that

$$\mathbf{N}_T|\mathbf{N}_t \sim N\left(\boldsymbol{\mu}_N\left(\mathbf{N}_t,\tau\right),\mathbf{S}_N\left(\tau\right)\right)$$

where

$$\begin{aligned}\boldsymbol{\mu}_N\left(\mathbf{N}_t,\tau\right) &= \boldsymbol{\Psi}\left(\tau\right)\mathbf{N}_t+\boldsymbol{\zeta}\left(\tau\right)\\ \mathbf{S}\left(\tau\right) &= \int_0^\tau \boldsymbol{\Psi}\left(\tau-s\right)\boldsymbol{\Sigma}_N\boldsymbol{\Sigma}_N'\boldsymbol{\Psi}\left(\tau-s\right)'ds\\ \boldsymbol{\zeta}\left(\tau\right) &= \int_0^\tau \boldsymbol{\Psi}\left(\tau-s\right)\mathbf{A}_N ds\end{aligned}$$

and

$$\boldsymbol{\Psi}\left(\tau\right)=\mathbf{U}\exp\left(\boldsymbol{\Lambda}\cdot\tau\right)\mathbf{U}^{-1}$$

where,since $\mathbf{B}_N$ has real and distinct eigenvalues, $\boldsymbol{\Lambda}$ is the diagonal matrix with its eigenvalues on the principal diagonal, $\mathbf{U}$ is the matrix of the associated eigenvectors, and $\exp\left(\boldsymbol{\Lambda}\cdot T\right)$ is the diagonal matrix with $e^{\lambda_i T}$ in its $ii-$th position. Direct computations yields

$$\boldsymbol{\Psi}\left(\tau\right)=\begin{pmatrix}1 & \left(\gamma-1\right)\left(1-e^{-\alpha\tau}\right) & \psi_{13}\left(\tau\right) & \beta_h\left(e^{-k_h\tau}-1\right)\\ 0 & e^{-\alpha\tau} & \frac{1}{\alpha-k}\left(e^{-k\tau}-e^{-\alpha\tau}\right) & 0\\ 0 & 0 & e^{-k\tau} & 0\\ 0 & 0 & 0 & e^{-k_h\tau}\end{pmatrix}$$

with

$$\psi_{13}\left(\tau\right)=\frac{\left(\gamma-1\right)e^{-\alpha\tau}}{\alpha-k}-\frac{\left(\gamma k-\alpha-\beta k+\beta\alpha\right)e^{-k\tau}}{\left(\alpha-k\right)k}+\frac{\beta-1}{k}$$

Proof: See e.g. Duffie (appendix). The computation of $\boldsymbol{\Psi}\left(\tau\right)$ from the formula is tedious, but straighforward.[3]

[3] The lemma also shows that

$$E_t\left[y_{t+\tau}\right] = \mu_2\left(\tau\right)=k\overline{\theta}\int_0^\tau \frac{1}{\lambda-k}\left(e^{-k(\tau-s)}-e^{-\lambda(\tau-s)}\right)ds$$

- From this Lemma, the price of the asset is simply

$$S_t = \frac{1}{\pi_t} E_t \left[\int_t^\infty e^{-\phi\tau + z_\tau} d\tau \right]$$

- The Lemma implies that

$$z_\tau \sim \mathcal{N}\left(\mu_1 \left(N, \tau\right), S_1\left(\tau\right)\right)$$

with

$$\begin{aligned} \mu_1(\tau) &= z_t + (\gamma - 1)\left(1 - e^{-\alpha\tau}\right) y_t + \psi_{13}(\tau)\,\theta_t \\ &\quad + \beta_h\left(e^{-k_h\tau} - 1\right) h_t + k\overline{\theta}\psi_{13}^I(\tau) \end{aligned}$$

with

$$\psi_{13}^I(\tau) = \frac{(\gamma - 1)}{\alpha - k} Q_\lambda(\tau) - \frac{(\gamma k - \alpha - \beta k + \beta\alpha)}{(\alpha - k)\,k} Q_k(\tau) + \frac{\beta - 1}{k}\tau$$

- Appealing to Fubini's theorem, we can compute easily the price of the asset from this result: In fact we obtain

$$\begin{aligned} S_t &= \frac{1}{\pi_t}\int_t^\infty e^{-\phi\tau} E_t\left[e^{z_\tau}\right] d\tau \\ &= \frac{1}{\pi_t}\int_t^\infty e^{-\phi\tau} e^{\mu_1(\tau - t) + \frac{1}{2}S_1(\tau - t)} d\tau \\ &= e^{-(1-\gamma)y_t + c_t + z_t}\int_t^\infty e^{(\gamma-1)\left(1 - e^{-\alpha\tau}\right)y_t + \psi_{13}(\tau)\theta_t + \beta_h\left(e^{-k_h\tau} - 1\right)h_t + Q_0(\tau)} d\tau \\ &= e^{\beta c_t + h_t}\int_t^\infty e^{(\gamma-1)(1 - e^{\alpha\tau})y_t + \psi_{13}(\tau)\theta_t + \beta_h\left(e^{-k_h\tau} - 1\right)h_t + Q_0(\tau)} d\tau \end{aligned}$$

$$\begin{aligned} &= k\overline{\theta}\frac{1}{\lambda - k}\left(\frac{1 - e^{-k\tau}}{k} - \frac{1 - e^{-\lambda\tau}}{\lambda}\right) \\ &\longrightarrow \frac{k\overline{\theta}}{\lambda - k}\left(\frac{1}{k} - \frac{1}{\lambda}\right) = \frac{1}{\lambda}\overline{\theta} \end{aligned}$$

with

$$Q_0(\tau) = -\phi\tau + k\overline{\theta}\psi_{13}^I(\tau) + \frac{1}{2}S_1(\tau)$$

- Finally, we can write simply

$$S_t = D_t G(y_t, \theta_t, h_t)$$

with

$$G(y_t, \theta_t, h_t) = \int_t^\infty e^{(\gamma-1)\left(1-e^{-\alpha\tau}\right)y_t + \psi_{13}(\tau)\theta_t + \beta_h\left(e^{-k_h\tau}-1\right)h_t + Q_0(\tau)}d\tau$$

- Below, I calibrate the economy to various β's and β_h's

2.3 The Risk Free Rate

- Before moving on, it is interesting to see the implications for the riskless rate: we have

$$\begin{aligned} d\pi_t/\pi_t &= -\phi dt - \gamma dy_t + \frac{1}{2}\gamma^2 dy^2 - dx_t \\ &= -\phi dt - \gamma\left((\theta_t - \alpha y_t)dt + \boldsymbol{\sigma}_c d\mathbf{B}_t\right) \\ &\quad + \frac{1}{2}\gamma^2\boldsymbol{\sigma}_c\boldsymbol{\sigma}_c' - \alpha(c_t - x_t)dt \\ &= \left(-\phi - \gamma(\theta_t - \alpha y_t) + \frac{1}{2}\gamma^2\boldsymbol{\sigma}_c\boldsymbol{\sigma}_c' - \alpha(c_t - x_t)\right)dt \\ &\quad - \gamma\boldsymbol{\sigma}_c d\mathbf{B}_t \end{aligned}$$

- yielding

$$\begin{aligned} r_t &= -E_t\left[\frac{d\pi_t}{\pi_t}\right] = \phi + \gamma\left(\theta_t - \alpha y_t\right) + \alpha y_t - \frac{1}{2}\gamma^2 \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c' \\ &= \phi + \gamma\theta_t + \left(1-\gamma\right)\alpha y_t - \frac{1}{2}\gamma^2 \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c' \end{aligned}$$

- That is, a high y_t implies a low interest rate r_t when $\gamma > 1$.
- The intuition is as follows: Recall that

$$y_t = c_t - x_t = \log\left(\frac{C_t}{X_t}\right)$$

- The representative agent here wants to smooth out C_t/X_t, that is, the consumption over habit, as its preferences are defined on this ratio.
- Thus, if today C_t/X_t is high it means that this ratio will be lower in the future, due to mean reversion.
- This implies the agent wants to decrease consumption today relative to habit X_t. That is, it must increase savings, driving the interest rate r_t down.

- From Lemma 1, since $\left(\theta_T, y_T\right)|_{(\theta_t, y_t)} \sim N\left(\boldsymbol{\mu}_{2,3}\left(\tau\right), \mathbf{S}_{23}\left(\tau\right)\right)$ we obtain that its unconditional value is

$$\begin{aligned} E\left[r_t\right] &= \phi + \gamma E\left[\theta_t\right] + \left(1-\gamma\right)\alpha E\left[y_t\right] - \frac{1}{2}\gamma^2 \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c' \\ &= \phi + \gamma E\left[\theta_t\right] + \left(1-\gamma\right) E\left[\theta_t\right] - \frac{1}{2}\gamma^2 \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c' \\ &= \phi + E\left[\theta_t\right] - \frac{1}{2}\gamma^2 \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c' \end{aligned}$$

- This formula gives hope to find a low interest rate.

- What about its time variation? We have

$$dr_t = \gamma d\theta_t + (1-\gamma)\,\alpha dy_t$$

- So that the diffusion part of the interest rate is given by

$$\sigma_r = \gamma\sigma_\theta + (1-\gamma)\,\alpha\sigma$$

- Notice that because $d\theta_t$ and dy_t are perfectly correlated, we can set the instantaneous volatility of the interest rate to zero by choosing appropriately the preference parameter γ and α.
- In particular

$$\sigma_r = 0 = \gamma\sigma_\theta + (1-\gamma)\,\alpha\sigma_c$$

- That is, by choosing

$$\alpha = \frac{\gamma\sigma_\theta}{(\gamma-1)\,\sigma_c}$$

  we can make the local volatility of the interest rate zero.
- Yet, the interest rate is not constant. In fact, we only have achieved that is local volatility is zero, in the sense that it is a process without Brownian shocks.

- However, from Lemma 1, we know that unconditionally, r_t is normally distributed. Let $\mathbf{S}_{ij} = \lim_{\tau\to\infty}[\mathbf{S}(t)]_{ij}$, where $\mathbf{S}(t)$ is given in Lemma 1 (this can be computed explicitly).

- Then, we have that the unconditional distribution of r_f is

$$r_f \sim \mathcal{N}\left(E\left[r_f\right], \gamma^2\mathbf{S}_{33} + (1-\gamma)^2\alpha^2\mathbf{S}_{22} + 2\gamma(1-\gamma)\alpha\mathbf{S}_{23}\right)$$

- Unfortunately, as we shall see, to match the equity premium we need a large γ, which will lead to a large distribution for r_f, the typical problem of habit formation models. Notice that also $\mathbf{S}_{22}$ and $\mathbf{S}_{23}$ depend on α, so the distribution is in fact rather complex.[4]

2.4 The Price of a Claim to Consumption

- The price of a contingent claim that produces consumption is given by the special parametrization $\beta = 1$ and $\beta_h = 0$,

[4] In fact, it is easy to see that (for instance)

$$\begin{aligned}
S_{23}(\tau) &= \int_0^\tau (\mathbf{\Psi}(\tau-s)\mathbf{\Sigma}_N)^2 ds \\
&= \int_0^\tau e^{-2\lambda(\tau-s)}\sigma_c^2 ds + \left(\frac{1}{\lambda-k}\right)^2 \sigma_\theta^2 \int_0^\tau \left(e^{-k(\tau-s)} - e^{-\lambda(\tau-s)}\right)^2 ds \\
&\quad +2\frac{1}{\lambda-k}\int_0^\tau e^{-\lambda(\tau-s)}\left(e^{-k(\tau-s)} - e^{-\lambda(\tau-s)}\right)\sigma_\theta\sigma_c ds \\
&= Q_{2\lambda}(\tau)\sigma_c^2 + \left(\frac{1}{\lambda-k}\right)^2\sigma_\theta^2\left(Q_{2k}(\tau) + Q_{2\lambda}(\tau) - 2Q_{(k+\lambda)}(\tau)\right) \\
&\quad +2\frac{\sigma_\theta\sigma_c}{\lambda-k}\left(Q_{(\lambda+k)}(\tau) - Q_{2\lambda}(\tau)\right)
\end{aligned}$$

As $\tau \to \infty$ we find

$$\begin{aligned}
S_{23} &= \frac{1}{2\lambda}\sigma_c^2 + \left(\frac{1}{\lambda-k}\right)^2\sigma_\theta^2\left(\frac{1}{2k} + \frac{1}{2\lambda} - 2\frac{1}{k+\lambda}\right) \\
&\quad +2\frac{\sigma_\theta\sigma_c}{\lambda-k}\left(\frac{1}{\lambda+k} - \frac{1}{2\lambda}\right)
\end{aligned}$$

yielding

$$S_t = C_t G(y_t, \theta_t)$$

- with

$$G(y_t, \theta_t) = \int_0^\infty e^{(\gamma-1)\left(1-e^{-\alpha\tau}\right)y_t+\psi_{13}(\tau)\theta_t+Q_0(\tau)} d\tau$$

- and

$$\begin{aligned}
\psi_{13}(\tau) &= \frac{(\gamma-1)\left(e^{-\alpha\tau}-e^{-k\tau}\right)}{\alpha-k} < 0 \text{ for all } \gamma > 1. \\
Q_0(\tau) &= -\phi\tau + k\overline{\theta}\psi_{13}^I(\tau) + \frac{1}{2}S_1(\tau) \\
\psi_{13}^I(\tau) &= \frac{(\gamma-1)}{\alpha-k}\left(Q_\alpha(\tau) - Q_k(\tau)\right)
\end{aligned}$$

2.4.1 The Equity Premium and Volatility

- To obtain the equity premium, use Ito's Lemma

$$dS_t = GdC_t + CdG = S_t \left(\mu_S + \boldsymbol{\sigma}_S d\mathbf{B}\right)$$

- where

$$\boldsymbol{\sigma}_S = \left(1 + G_y\left(y_t, \theta_t\right)\right)\boldsymbol{\sigma}_c + G_\theta\left(y_t, \theta_t\right)\boldsymbol{\sigma}_\theta$$

- where

$$\begin{aligned} G_y\left(y_t, \theta_t\right) &= \frac{1}{G}\frac{\partial G}{\partial y} \\ &= \frac{\left(\gamma - 1\right)\int_0^\infty \left(1 - e^{-\lambda\tau}\right) e^{(\gamma-1)\left(1-e^{\lambda\tau}\right)y_t + \psi_{13}(\tau)\theta_t + Q_0(\tau)} d\tau}{G} \qquad (2) \\ G_\theta\left(y_t, \theta_t\right) &= \frac{1}{G}\frac{\partial G}{\partial \theta} \\ &= \frac{\int_0^\infty \psi_{13}\left(\tau\right) e^{(\gamma-1)\left(1-e^{\lambda\tau}\right)y_t + \psi_{13}(\tau)\theta_t + Q_0(\tau)} d\tau}{G} < 0 \qquad (3) \end{aligned}$$

- Thus, since

$$d\pi_t/\pi_t = -r_t dt - \gamma\boldsymbol{\sigma}_c d\mathbf{B}$$

- we obtain

$$
\begin{aligned}
E[dR] &= -cov\left(dR_t^i, \frac{d\pi_t}{\pi_t}\right) \\
&= \gamma\left(\left(1+G_y(y_t,\theta_t)\right)\boldsymbol{\sigma}_c\boldsymbol{\sigma}_c' + G_\theta(y_t,\theta_t)\,\boldsymbol{\sigma}_\theta\boldsymbol{\sigma}_c'\right)
\end{aligned}
$$

- The first adjustment is positive, but the second is negative, as $G_\theta(y_t,\theta_t) < 0$.
- However, we expect $G_y(y_t,\theta_t)$ to be "big" while we expect $G_\theta(y_t,\theta_t)$ to be small.

- Similarly, the volatility of the stock is simply given by

$$\sigma_S = \left(1+G_y(y_t,\theta_t)\right)\sigma_c + G_\theta(y_t,\theta_t)\,\sigma_\theta$$

- Under the condition of perfect correlation between $d\theta$ and dc, we have that the Sharpe ratio is constant and given by

$$SR_t = \frac{E[dR]}{\sigma_S} = \gamma\sigma_c$$

- In addition, we also have a little predictability of stock returns. We can re-write the expected returns formula as

$$E[dR] = \gamma\boldsymbol{\sigma}_c\boldsymbol{\sigma}_c' + \beta_{\text{pred}}(y_t,\theta_t)\,\frac{C}{S}$$

where

$$\beta_{\text{pred}}(y_t,\theta_t) = G \times \left(G_\theta\boldsymbol{\sigma}_\theta\boldsymbol{\sigma}_c' + G_y\boldsymbol{\sigma}_c\boldsymbol{\sigma}_c'\right)$$

2.4.2 Calibration

- We can estimate the process for consumption by a simple application of Kalman filter, obtaining the values

Table I: Estimates of the Parameters of the Model

$\overline{\theta}$	k	σ_C	σ_θ
0.0221	1.2325	0.0090	0.0046

- Notice that consumption growth has a lower volatility than the one usually obtained, about 1.2%. Indeed, even in my sample we have $std\left(\Delta \log\left(C\right)\right) = 1.11\%$. The lower value that I obtain is due to the assumed time variation in consumption growth θ_t which partly decreases the estimated instantaneous volatility of consumption.
- Of course, this makes the resolution of the equity premium puzzle even more challenging, and even higher coefficients of risk aversion than the usual ones have to be used.
- The following table shows a calibration exercise, for several parameters for the intertemporal discount ϕ and the coefficient of risk aversion γ.[5] The speed of mean reversion α has

[5] One may wonder whether γ is indeed the coefficient of relative risk aversion in this setting. In fact, it is, as the value function is given by

$$\begin{aligned} J\left(W_t, y_t\right) &= E\left[\int_t^\infty e^{-\phi(\tau-t)} \frac{C_\tau^{1-\gamma}/X_\tau^{1-\gamma}}{1-\gamma}\right] = \frac{1}{1-\gamma} E\left[\int_t^\infty e^{-\phi(\tau-t)} e^{(1-\gamma)y_\tau}\right] \\ &= \frac{1}{1-\gamma}\pi_t P_t e^{\phi t} = \frac{1}{1-\gamma} X^{(\gamma-1)} C_t^{-\gamma} P_t \end{aligned}$$

From $P_t = C_t G\left(y_t, \theta_t\right)$ we have $C_t^{-\gamma} = P_t^{-\gamma} G^\gamma$ yielding the value function

$$J\left(W_t, X_t, y_t\right) = \frac{1}{\gamma - 1}\left(\frac{W}{X_t}\right)^{1-\gamma} G\left(y_t\right)^\gamma$$

been chosen to ensure that the local volatility of the risk free rate is zero.

Table II: Calibration Exercise

ϕ	γ	α	r_f	$\sigma(r_f)$	$E[dR]$	σ_S	SR	$\overline{PD}$	β_{pred}
0.05	2.00	1.02	0.07	0.00	0.00	0.02	0.02	19.86	0.00
0.05	12.00	0.56	0.06	0.04	0.01	0.10	0.11	20.00	0.17
0.05	22.00	0.54	0.03	0.08	0.03	0.18	0.20	20.40	0.64
0.05	32.00	0.53	-0.01	0.12	0.07	0.25	0.29	21.07	1.44
0.05	42.00	0.52	-0.07	0.16	0.13	0.33	0.38	22.05	2.65
0.05	52.00	0.52	-0.15	0.19	0.19	0.41	0.47	23.37	4.35
0.07	2.00	1.02	0.09	0.00	0.00	0.02	0.02	14.27	0.00
0.07	12.00	0.56	0.08	0.04	0.01	0.09	0.11	14.37	0.12
0.07	22.00	0.54	0.05	0.08	0.03	0.17	0.20	14.65	0.44
0.07	32.00	0.53	0.01	0.12	0.07	0.24	0.29	15.12	0.99
0.07	42.00	0.52	-0.05	0.16	0.12	0.32	0.38	15.80	1.81
0.07	52.00	0.52	-0.13	0.19	0.19	0.40	0.47	16.73	2.97
0.09	2.00	1.02	0.11	0.00	0.00	0.02	0.02	11.11	0.00
0.09	12.00	0.56	0.10	0.04	0.01	0.09	0.11	11.18	0.09
0.09	22.00	0.54	0.07	0.08	0.03	0.16	0.20	11.39	0.33
0.09	32.00	0.53	0.03	0.12	0.07	0.23	0.29	11.75	0.73
0.09	42.00	0.52	-0.03	0.16	0.12	0.31	0.38	12.27	1.34
0.09	52.00	0.52	-0.11	0.19	0.18	0.38	0.47	12.98	2.20

- The implication of this analysis is that in order to match at the same time equity premium, a low interest rate, we need to choose the speed of mean reversion in habit appropriately.

- Notice that we still need a high coefficient of relative risk aversion, $\gamma = 32$ for instance, and thus we do not solve the traditional puzzle.
- Yet, the model is able to deliver simultaneously a high equity risk premium, and a low interest rate, with habit formation. Instead, the distribution of interest rates is too large.
- Yet, the interest rate moves too much: the reason is that after good news

2.5 Individual Securities

- We can use a similar methodology to obtain the volatility and expected returns of individual securities.
- From the price of the asset, we can obtain its volatility function

$$\boldsymbol{\sigma}_S^i = \boldsymbol{\sigma}_D + G_y \boldsymbol{\sigma}_c + G_\theta \boldsymbol{\sigma}_\theta + G_h \boldsymbol{\sigma}_h \tag{4}$$

- Thus, the expected return of asset i is given by

$$E_t\left[dR_t^i\right] = \gamma\left(\boldsymbol{\sigma}_D + G_y \boldsymbol{\sigma}_c + G_\theta \boldsymbol{\sigma}_\theta + G_h \boldsymbol{\sigma}_h\right) \boldsymbol{\sigma}_c'$$

- A conditional CAPM representation holds.
- Note that the claim to consumption is perfectly correlated with consumption growth, and thus with the pricing kernel π_t.
- Indeed, recall that the volatility of the market is given by

$$\sigma_S^M = \sigma_c + G_y^M \sigma_c + G_\theta^M \sigma_\theta \tag{5}$$

- (the superscript "M" denotes "market".

- Since we can write

$$E_t\left[dR_t^i\right] = \left(\beta_h\sigma_h + \beta\sigma_c + G_y\sigma_c + G_\theta\sigma_\theta + G_h\sigma_h\right)\gamma\sigma_c$$

- and the fact that

$$E\left[dR_t^M\right] = \left(\sigma_c + G_y^M\sigma_c + G_\theta^M\sigma_\theta\right)\gamma\sigma_c$$

- we have, simply

$$\gamma\sigma_c = \frac{1}{\left(\sigma_c + G_y^M\sigma_c + G_\theta^M\sigma_\theta\right)}E_t\left[dR_t^M\right]$$

- This we can be substituted into the previous expression, to find

$$E_t\left[dR_t^i\right] = \beta_R^i\left(y_t, \theta_t, h_t\right)E_t\left[dR_t^M\right] \tag{6}$$

- where the beta is

$$\begin{aligned}\beta_{R,t}^i &= \frac{\beta_h\sigma_h + \beta\sigma_c + G_y\sigma_c + G_\theta\sigma_\theta + G_h\sigma_h}{\sigma_c + G_y^M\sigma_c + G_\theta^M\sigma_\theta} \\ &= \frac{cov\left(dR_t^i, dR_t^M\right)}{var\left(dR_t^m\right)}\end{aligned}$$

- This finding generates interesting questions:
  1. For a plausible calibration, what is the variation in $\beta_{R,t}^i$?
  2. There is a very large literature about conditional CAPM models. Does this model fit the predictions?
  3. Indeed, does this model imply that the unconditional CAPM fails?

- As for the last question, recall that even if the CAPM holds conditionally, it may not hold unconditionally.

- This can be seen immediately from the conditional CAPM representation (6): If we condition down (i.e. take unconditional expectation on the LHS and RHS, we obtain

$$E\left[E_t\left[dR_t^i\right]\right] \;=\; E\left[\beta_{R,t}^i E_t\left[dR_t^M\right]\right]$$

or

$$E\left[dR_t^i\right] \;=\; E\left[\beta_{R,t}^i\right]\; E\left[dR_t^M\right] + Cov\left[\beta_{R,t}^i\; E_t\left[dR_t^M\right]\right]$$

- The $\beta_{R,t}^i$ depends on the same state variables that affect the conditional expectation $E_t\left[dR_t^M\right]$.

- Thus, the covariance term on the RHS is non-zero, and the unconditional CAPM does not work.

- The above questions are interesting, but I am not aware of any study that actually tackles them.

2.6 External Habit: Campbell and Cochrane (1999).

- A problem with previous setting is that the interest rate moves too much.
- Remember that this is due to the time variation in $\log(C_t/X_t)$ which the agent wants to smooth out.
  - E.g. if $\log(C_t/X_t)$ is too low, the agent expects it to increase in the future. This requires an increase in consumption today, which increases borrowing and thus the interest rate
- Since we need a large variation $\log(C_t/X_t)$ to generate a sizable equity premium, we are in trouble.
- The problem is that as $\log(C_t/X_t)$ gets small, there is no counteracting force that stops the agent from borrowing.
- This is due to the particular specification of preferences, in which habit X_t enters as a ratio.
- Other specifications lead to additional terms, that restrict the agent willingness of borrowing when C_t gets too close to X_t.

- Constantinides (1990), Detemple and Zapatero (1991), Campbell and Cochrane (1999) and others use the following representation of habit. The representative agent maximizes

$$E\left[\int_0^\infty u\left(C_t, X_t, t\right) dt\right], \tag{7}$$

- where the instantaneous utility function is give by

$$u\left(C_t, X_t, t\right) = \begin{cases} e^{-\rho t} \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 1 \\ e^{-\rho t} \log\left(C_t - X_t\right) & \text{if } \gamma = 1 \end{cases} \tag{8}$$

- This is the *difference* model.
- Intuitively, as $C_t - X_t$ decreases, the agent is approacing negative infinity utility.
- This will increase precautionary savings (buy bonds to decrease the probability that C_t hits X_t), which in turn decreases the interest rate r_t.
- This effect goes against the intertemporal smoothing effect discussed earlier, and it may stabilize the interest rate.
- However, the model above is not homogeneus, and thus it is hard to work with.
- One additional problem is that in endowment economies it is not possible to guarantee that $C_t > X_t$ all the time, for standard specification of habit X_t.

- For instance, assume that consumption (endowment) follows the geometric Brownian motion

$$\frac{dC_t}{C_t} = \mu_c dt + \sigma_c dB_t$$

- Assume also X_t is just a weighted average of past consumption:

$$X_t = X_0 e^{-\alpha t} + \alpha \int_0^t e^{-\alpha(\tau - t)} C_\tau d\tau$$

- As we have seen

$$dX = \alpha \left(C_t - X_t \right) dt$$

- Consider now the following quantity

$$S_t = \frac{C_t - X_t}{C_t} \tag{9}$$

- Campbell and Cochrane (1999) call S_t the *Surplus Consumption Ratio*: It is the percentage difference of consumption above habit. We return on the interpretation later.

- Clearly, we must have $S_t \in [0, 1]$

- However, if we apply Ito's Lemma to S_t we find

$$dS_t = \left(-\alpha S_t + \left(S_t - 1 \right) \sigma_c^2 \right) dt + \left(1 - S_t \right) \frac{dC_t}{C_t}$$

- or, equivalently,

$$dS = k\left(\overline{S} - S_t\right) dt + \lambda\left(S_t\right) dB_t \tag{10}$$

- where

$$\begin{aligned} k &= \mu_c - \alpha - \sigma_c^2 \\ \overline{S} &= \left(\mu_c - \sigma_c^2\right)/k \\ \lambda\left(S_t\right) &= \left(1 - S_t\right) \end{aligned}$$

- Note that S_t is:
  - Mean reverting: This is a consequence of habit formation and the fact that X_t is slow moving.
  - Perfectly correlated with innovations to consumption growth, given by dB_t.
  - The volatility of surplus is time varying.
- Note also that S_t is bounded above: when S_t reaches 1, the diffusion disappears, and the drift is negative. Thus, S_t is dragged down.
- However, nothing stops S_t from going below zero: When $S_t = 0$, the diffusion is still positive (and in fact large). Although the drift is also positive under the sensible assumption that $\mu_c > \sigma_c^2$, there is a non-zero probability that $S_t \leq 0$.
- This event of course is inconsistent with the preference specification.

- Campbell and Cochrane (1999) had a great intuition: Let's specify (10) for *log* surplus $s_t = \log(S_t)$, and specify $\lambda(s_t)$ in a way to ensure $S_t = exp(s_t) \in [0, 1]$.
- In addition, they specified $\lambda(s_t)$ to obtain specific properties of the interest rate process r_t (e.g. constant!)
- Unfortunately, their specification does not yield closed form solutions for prices.
- I therefore follow Santos and Veronesi (2005) (in progress), which generalizes the setting in Menzly, Santos, Veronesi (2004) to the power utility case.
- All of the effects that Campbell and Cochrane (1999) talk about in their paper show up in our setting too.
- To introduce the methodology, consider first the stochastic discount factor implied by the model

$$\pi_t = e^{-\rho t} \frac{\partial u\left(C_t, X_t\right)}{\partial C_t} = e^{-\rho t}\left(C_t - X_t\right)^{-\gamma} = e^{-\rho t} C_t^{-\gamma} S_t^{-\gamma}$$

- The surplus consumption ratio acts as a "preference shock", as it changes the curvature of the utility function, given by γS_t^{-1}.
- In other words, it induces a time variation in risk preferences of the representative agent.
- Starting with the observation that the surplus is mean reverting, Campbell and Cochrane (1999) consider the particular monotonic transformation $s_t = \log(S_t)$ and model s_t as mean reverting.

- Santos and Veronesi (2005) use a different monotonic transformation, namely
$$G_t = S_t^{-\gamma} \tag{11}$$

- Assume then that G_t is mean reverting
$$dG_t = k\left(\overline{G} - G_t\right)dt - \alpha\left(G_t - \lambda\right)\sigma_c dB_t \tag{12}$$

- Note the following:
  1. G_t is mean reverting, like S_t.
  2. G_t is *negatively* perfectly correlated with innovations to consumption dB_t.
  3. G_t is bounded below by $\lambda > 1$. That is, we restrict $C_t > X_t$ at all times.
- These are the same properties of Campbell and Cochrane (1999).

2.6.1 Results

- Let's start with the interest rate. Since $\pi_t = e^{-\rho t}C_t^{-\gamma}G_t$ we obtain immediately that the SDF follows the process
$$\frac{d\pi_t}{\pi_t} = -r_t^f dt - \sigma_\pi dB_t,$$
- where

$$r_t^f = \rho + \gamma\mu_c - \frac{1}{2}\gamma(\gamma+1)\sigma_c^2 + k\left(1-\overline{G}S^\gamma\right) - \gamma\alpha\left(1-\lambda S^\gamma\right)\sigma_c^2 \quad (13)$$

- and

$$\sigma_\pi = \gamma + \alpha\left(1-\lambda S_t^\gamma\right)\sigma_c. \quad (14)$$

- **Comments:**
  1. The first three terms in r_t are standard.
  2. The fourth term $k\left(1-\overline{G}S^\gamma\right)$ represents the intertemporal substitution effect already discussed for Abel (1990)

$$\text{Low } S_t \;\rightarrow\; \text{high expected } S_\tau \text{ in future } \rightarrow$$
$$\rightarrow\; \text{Borrow to increase } C_t \;\rightarrow\; r_t \text{ high}$$

  3. The last term $-\gamma\alpha\left(1-\lambda S^\gamma\right)$ represents an additional precautionary savings term:

$$\text{Low } S_t \;\rightarrow\; \text{higher probability } C_\tau = X_\tau \text{ in the future } \rightarrow$$
$$\rightarrow\; \text{Save more today } \rightarrow\; r_t \text{ low}$$

  4. Campbell and Cochrane (1999) choose parameters so that these two effects cancel each other → constant r_t
     - We could do the same here, but unfortunately the model does not have enough flexibility.
     - If we exactly fix r_t we won't be able to match other moments of returns

5. The volatility of the stochastic discount factor now depends on S_t^γ

   Low S_t $\rightarrow$ higher curvature of the utility function γS_t^{-1} $\rightarrow$
   $\rightarrow$ Higher aversion to risk $\rightarrow$ Higher price of risk

- Coming to the stock price of a consumption claim, we have

$$P_t = E_t \left[\int_t^\infty \left(\frac{\pi_\tau}{\pi_t} \right) C_\tau d\tau \right] \qquad (15)$$

- Substituting, we obtain

$$P_t = C_t^\gamma S^\gamma E_t \left[\int_t^\infty e^{-\rho(\tau - t)} C_\tau^{1-\gamma} G_\tau d\tau \right] \qquad (16)$$

- It could be useful to show the steps to obtain a closed form solution (the details are in the appendix)

  1. Define
  $$M_t = C_t^{1-\gamma} G_t \text{ and } \quad N_t = C_t^{1-\gamma}$$

  2. Use Ito's Lemma to obtain the law of motion of both M_t and N_t. It turns out that if we define $\mathbf{Z}_t = (M_t, N_t)$, we can write
  $$d\mathbf{Z}_t = \mathbf{A}\mathbf{Z}_t dt + \mathbf{\Sigma}_t dB_t$$
  where $\mathbf{A}$ is explicitly given in the appendix.

3. This implies

$$E_t\left[\mathbf{Z}_\tau\right] = \mathbf{U}\mathbf{E}\left(\tau - t\right)\mathbf{U}^{-1}\mathbf{Z}_t$$

   where $\mathbf{U}$ is the matrix of eigenvectors of $\mathbf{A}$, and $\mathbf{E}\left(\tau\right)$ is the diagonal matrix with $\left[\mathbf{E}\left(\tau\right)\right]_{ii} = e^{\omega_i \tau - t}$ where ω_i is the eigenvalue of $\mathbf{A}$.

4. Thus

$$\begin{aligned} E_t\left[M_\tau\right] &= (1,0)\,\mathbf{U}\mathbf{E}\left(\tau - t\right)\mathbf{U}^{-1}\mathbf{Z}_t \\ &= \sum_{i=1}^{2}\sum_{k=1}^{2} u_{1i} e^{\omega_i(\tau - t)} u_{ik}^{-1}\mathbf{Z}_{k,t} \end{aligned}$$

5. Use Fubini, and this expectation to solve explicitly for the integral

$$\begin{aligned} P_t &= C_t^\gamma S_t^\gamma \int_t^\infty E_t\left[e^{-\rho(\tau - t)} M_\tau\right] d\tau \\ &= C_t^\gamma S_t^\gamma \sum_{i=1}^{2}\sum_{k=1}^{2} u_{1i}\left(\int_t^\infty E_t\left[e^{(\omega_i - \rho)(\tau - t)}\right] d\tau\right) u_{ik}^{-1}\mathbf{Z}_{k,t} \\ &= C_t^\gamma S_t^\gamma \sum_{k=1}^{2}\left(\sum_{i=1}^{2}\frac{u_{1i}u_{ik}^{-1}}{\rho - \omega_i}\right)\mathbf{Z}_{k,t} \\ &= C_t^\gamma S_t^\gamma\left(b_1 C_t^{1-\gamma} G_t + b_2 C_t^{1-\gamma}\right) \\ &= C_t\left(b_1 + b_2 S_t^\gamma\right) \end{aligned}$$

6. Finally, compute explicitly b_1 and b_2, which turn out to have simple formulas

$$b_1 = \frac{1}{\alpha_1}$$
$$b_2 = \frac{k\overline{G} + \alpha(1-\gamma)\lambda\sigma_c^2}{\alpha_1\alpha_2}$$

with

$$\alpha_1 = \rho - (1-\gamma)\mu_c + \frac{1}{2}(1-\gamma)\gamma\sigma_c^2 + k + \alpha(1-\gamma)\sigma_c^2$$
$$\alpha_2 = \rho - (1-\gamma)\mu_c + \frac{1}{2}(1-\gamma)\gamma\sigma_c^2$$

- The implications for the $P_t/C_t = b_1 + b_2 S_t^\gamma$ are obvious:
  - a higher surplus consumption ratio S_t translates in lower risk preference, and thus a higher price.
- Note also that intertemporal smoothing hits here too. From the form of b_1 and b_2, a high consumption growth μ_c translates into a lower P/C ratio, as we saw with learning.
  - Therefore, learning abour μ_c, for instance, will generate the same problem it did for the standard power utility case.
  - Indeed, the same was true for the Abel (1990) specification.

- What about the volatility and the equity premium?
- By using Ito's Lemma, we have

$$E_t\left[dR_t\right] = \left(\gamma + \alpha\left(1 - \lambda S_t^\gamma\right)\right)\sigma_R\left(S_t\right)\sigma_c$$

$$\sigma_R\left(S_t\right) = \left[1 + \frac{b_2 S_t^\gamma\left(1 - \lambda S_t^\gamma\right)\alpha}{b_1 + b_2 S_t^\gamma}\right]\sigma_c.$$

- How does this model performs?
- The following are some statistics of the market portfolio:

Table I
Basic moments

Panel A: Summary statistics for the market portfolio

$E(R^M)$	vol(R^M)	r^f	vol(r^f)
7.71%	16.25%	1.44%	3.08%

Panel B: Predictability regressions

	Panel B-1: Sample 1948-2001				Panel B-2: Sample 1948-1995			
Horizon	4	8	12	16	4	8	12	16
$\ln\left(\frac{D}{P}\right)$	.13	.2	.26	.35	.28	.48	.63	.78
t−stat.	(2.13)	(1.65)	(1.34)	(1.29)	(4.04)	(4.00)	(4.49)	(5.41)
R^2	.09	.10	.11	.14	.19	.32	.43	.54

- A simple calibration of the economy (not much parameter search here) is as follows:

Table III
Model parameters used in the simulation

Panel A: Consumption and preference parameters

μ_c	σ_c	γ	ρ	$\gamma/\overline{S}$	$\min\{\gamma/S_t\}$	α	k
.02	.015	1.5	.072	48	27.75	77	.13

Table IV
Basic moments in simulated data

Panel A: Summary statistics for the aggregate portfolio

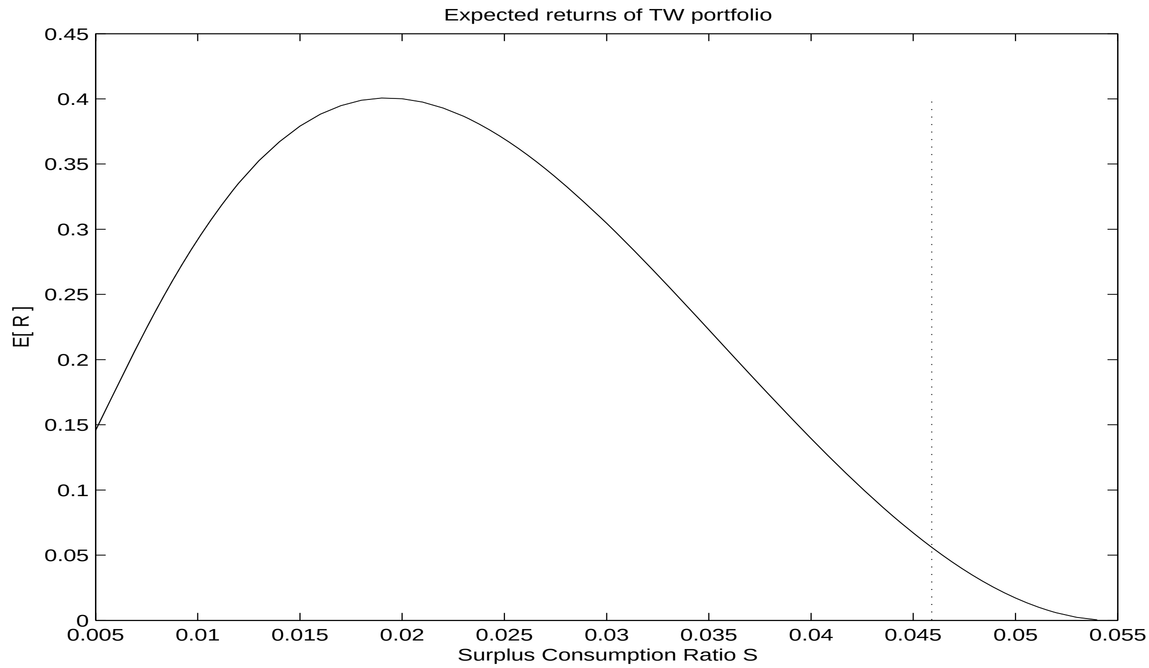
$E(R^M)$	$\text{vol}(R^M)$	r^f	$\text{vol}(r^f)$
9.96%	24.15%	.91%	5.41%

Panel B: Predictability regressions

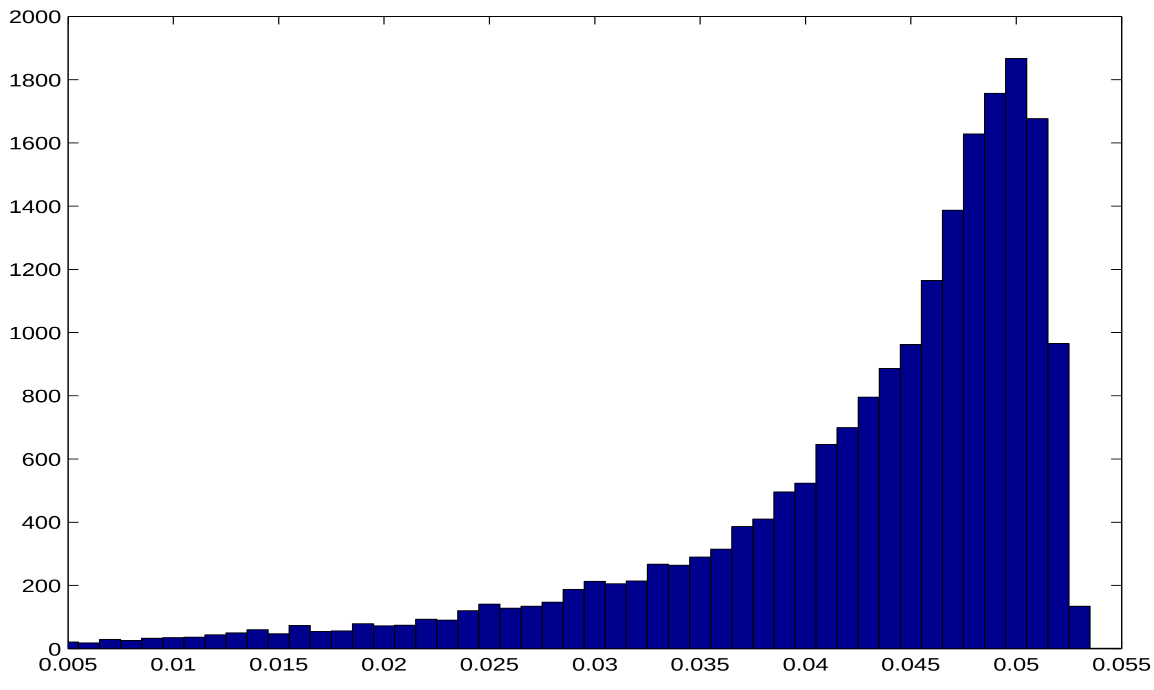
Horizon	4	8	12	16
$\ln\left(\frac{D}{P}\right)$	.73	.86	.88	.85
R^2	.25	.30	.29	.27

- The model does well, although the volatility of interest rates is a little too high
- The following Figures shows the source of the effects

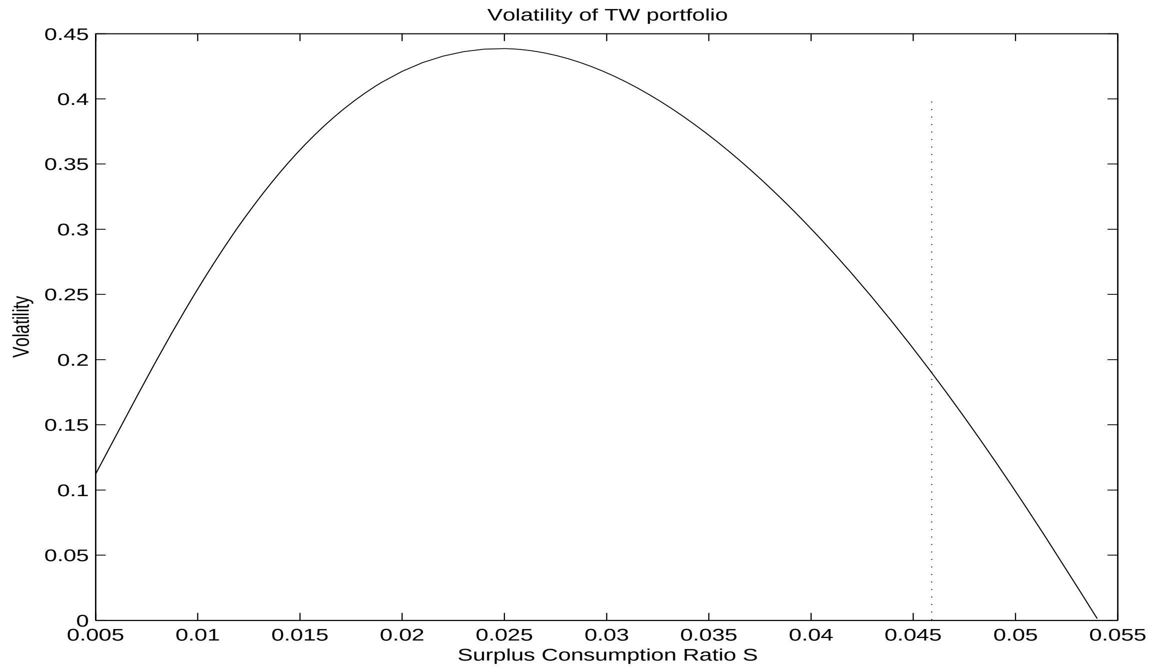
Expected Return



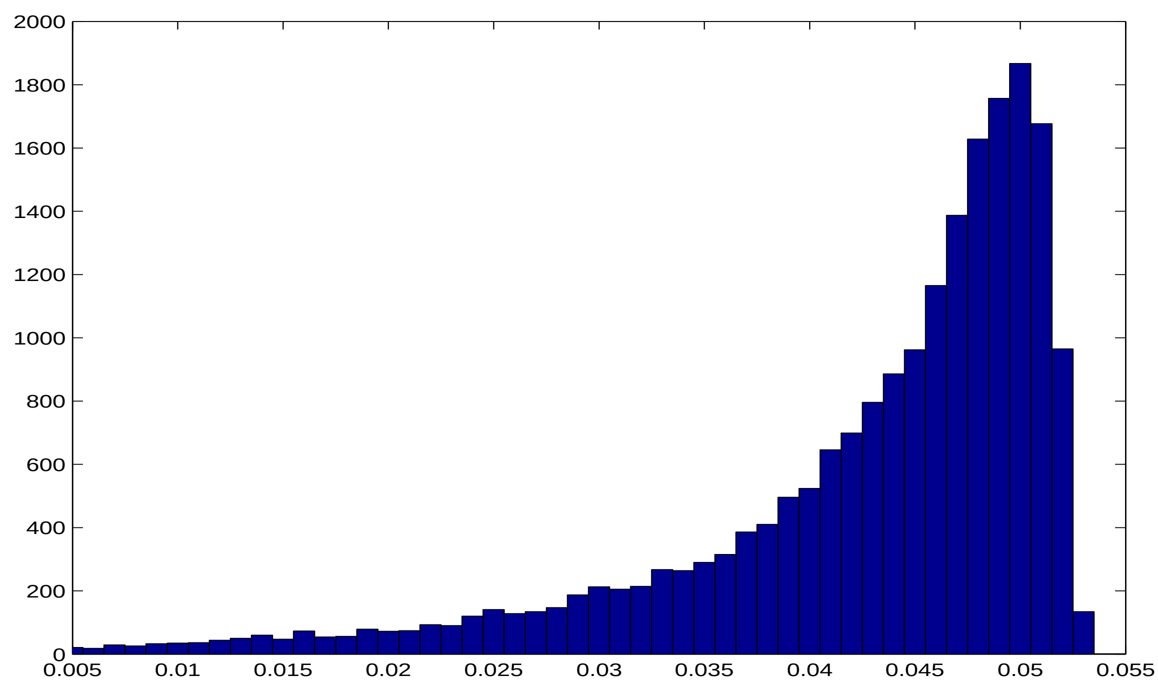
Distribution of S_t



Conditional Volatility



Distribution of S_t



- Menzly, Santos and Veronesi (2004, JPE) use a similar model (with $\gamma = 1$) to investigate the implications of time varying risk preferences for predictability of individual stocks, or the market in the presence of labor income.
  - As we will see, they find that when the representative agent has time varying risk preferences, as above, then time variation in the expected dividend growth of a security generates a *positive* relation between expected returns and price/dividend ratio.
- Santos and Veronesi (2005) takes after this intuition, and discuss the implications for the value spread puzzle.

2.7 Internal Habit:Detemple and Zapatero (1991).

- We now discuss briefly the Habit Formation models, still in the context of a standard Lucas economy.
- The model I refer to is by Detemple and Zapatero (1991).
- Consider a standard endowment economy as in TN4 with complete markets.
- Let security prices be driven by the usual Ito process

$$d\mathbf{S} = \mathbf{I}_S \boldsymbol{\mu}_t dt + \mathbf{I}_S \boldsymbol{\sigma}_t d\mathbf{B}_t$$

- Assume a risk-free asset following the Ito process r_t and define the market price of risk as

$$\boldsymbol{\nu}_t = \boldsymbol{\sigma}_t^{-1} \left(\boldsymbol{\mu}_t - r_t \mathbf{1}_n \right)$$

- Assume that an agent's endowment process e_t follows also an Ito process

$$de_t = \mu_{e,t} dt + \boldsymbol{\sigma}_{e,t} d\mathbf{B}_t$$

- Consumption processes c are defined in the usual space of square integrable processes and a portfolio process is given by $(\varphi^0, \boldsymbol{\varphi})$ of dollar amounts invested in bonds and stocks. Usual conditions apply.
- Preferences are described by the utility function

$$U(c, z) = E\left[\int_0^T e^{-\phi t} u(c_t, z_t)\, dt \right] \tag{17}$$

- where

$$z_t = z_0 e^{-\alpha t} + \delta \int_0^t e^{-\alpha(t-s)} c_s ds \tag{18}$$

- The index z_t represents the *standard of living* of the decision maker.
- Assume the usual regularity conditions to the instantaneous utility function $u(c_t, z_t)$. Namely,
  1. Increasing and strictly concave in c and satisfies Inada conditions for each z;
  2. Decreasing in z;
  3. Concave in (c, z);
- As usual, $\mathcal{I}_u(., z)$ represents the inverse of $u_c(., z)$.
- The index (18) has been introduced by Constantinides (1990) in the literature.
- He investigates a production economy.
- By Ito's lemma, we have

$$dz_t = (\delta c_t - \alpha z_t)\, dt$$

- Hence, current consumption increases the standard of leaving by δc_t and this decays at the rate α.
- Assuming $\delta = 0$ effectively the model collapses to an external habit formation model, where the "standard of living" is exogenous. Although in the present approach it would be deterministic ($z_t = z_0 e^{-\alpha t}$), it turns out that all the results hold if α is a stochastic process rather than a constant parameter.

- By assuming various forms of $u(c, z)$ one can recover other models. For example,

$$u(c, z) = \begin{cases} v(c - z) & \text{if} \quad c \geq z \\ -\infty & \text{if} \quad c < z \end{cases} \tag{19}$$

- has been proposed by Constantinides (1990), Campbell and Cochrane (1999) and others.
- Detemple and Zapatero (1991) must make additional assumptions to prove existence.
- Assume

$$u_c(e_t, z_t^e) + \delta E\left[\int_t^T e^{-(\phi+\alpha)(\tau-t)} u_z(e_\tau, z_\tau^e)\, d\tau\right] > 0$$

- where z_t^e is the standard of living resulting from the consumption of the aggregate endowment

$$z_t^e = e_0 e^{-\alpha t} + \delta \int_0^t e^{-\alpha(t-s)} e_s ds \tag{20}$$

2.8 Optimal Consumption

- Maximizing utility subject to standard budget constraints is known to be equivalent to the following (see TN4)

$$\max_{\left(c,\left(\varphi^0,\boldsymbol{\varphi}\right)\right)} U(c, z)$$

- subject to

$$E\left[\int_0^T \pi_t c_t dt\right] \leq E\left[\int_0^T \pi_t e_t dt\right]$$

- where

$$\pi_t = e^{\left(\int_0^t -\left(r_u + \frac{1}{2}\boldsymbol{\nu}_u \boldsymbol{\nu}'_u\right) du + \int_0^t \boldsymbol{\nu}'_u d\mathbf{B}_u\right)}$$

- As usual, define the Lagrangian

$$\mathcal{L}\left(c, z, \lambda\right) = E\left[\int_0^T e^{-\phi t} u\left(c_t, z_t\right) - \lambda\left(\pi_t c_t - \pi_t e_t\right) dt\right] \quad (21)$$

- The additional difficulty is that if we choose the consumption at time c_t, the whole path z_τ from $[t, T]$ is going to change.
- This must be taken into account!
- Since markets are complete, we can maximize the Lagrangian state by state (that is, for each $\omega \in \Omega$).
- Given $\omega \in \Omega$, the problem becomes deterministic and we can apply optimal control methods.
- This yields the FOC

$$u_c\left(c_t, z_t\right) + \delta E_t\left[\int_t^T e^{-(\phi+\alpha)(\tau - t)} u_v\left(c_\tau, z_\tau\right) d\tau\right] = \lambda e^{\phi t} \pi_t$$

- Clearly, the second term on the LHS is the effect stemming from the fact that an increase in consumption today would increase the standard of leaving tomorrow (and in the future), which it will make it hard to "beat."
- Hence, increasing consumption today has a partially counterbalancing effect because the agent anticipate the increase in standard of living.

- The parameter λ as usual is obtained by the budget constraint along the optimal path (omitted!)
- Heuristically, we can then conclude from the above that positive innovations to endowment would generate a lower reaction to consumption than when habit does not matter.
- In fact, now if the decision maker increases the consumption rate, he/she is going to "suffer" in the future because of the increase in standard of living.
- This interpretation was put forward in Constantinides (1990) and Sundaresan (1989) in the context of constant opportunity sets.
- Although it turns out this does not work in general, it does under the linear model specified earlier.
- Define by

$$g_t^* = \lambda^* e^{\phi t}\pi_t - \delta E_t \left[\int_t^T e^{-(\phi+\alpha)(\tau-t)} u_v\left(c_\tau^*, z_\tau^*\right) d\tau \right]$$

- the total marginal cost of a marginal increase in consumption.
- Then the optimal consumption must satisfy

$$u_c\left(c_t^*, z_t^*\right) = g_t^*$$

- We then have the following result (Theorem 4.2, Detemple and Zapatero (1991)): Under the assumption of a linear model as in equation (19) we have

1. The total marginal cost of consumption is given by
$$g_t^* = \lambda^* e^{\phi t} \pi_t \left(1 + \delta E_t \left[\int_t^T e^{-\int_t^\tau (r_u + \alpha - \delta) du} d\tau\right]\right)$$

2. The optimal consumption policy is given by
$$c_t^* = z_0 e^{(\delta-\alpha)t} + \mathcal{I}_v \left(g_t^*\right) + \delta \int_0^t e^{(\delta-\alpha)(t-\tau)} \mathcal{I}_v \left(g_\tau^*\right) d\tau$$
where $\mathcal{I}_v(.)$ is the inverse of the marginal utility $v(x)$ introduced in (19).

3. The standard of living index is given by
$$z_t^* = z_0 e^{(\delta-\alpha)t} + \delta \int_0^t e^{(\delta-\alpha)(t-\tau)} \mathcal{I}_v \left(g_\tau^*\right) d\tau$$

- Notice that g_t^* is given by the state-price density $\lambda^* e^{\phi t} \pi_t$ (as usual) adjusted by a factor that reflects the importance of habits in the investors' evaluation of future consumption streams.

- The idea is that the utility function $v(c-z)$ described in (19) gives infinite penalty when $c < z$. Hence, optimal consumption is chosen to give zero probability for this happening in the future.

- To ensure this, the optimal state-price density increases to induce a lower current spending (the marginal utility of consumption is decreasing with consumption).

- Notice that no habit formation ($\delta = 0$ and $z_0 = 0$) imply the usual results obtained in TN3 and TN4.

- If we only assume $\delta = 0$, then we are in the "external" habit formation case, in which case we obtain
$$c_t^* = z_0 e^{-\alpha t} + \mathcal{I}_v \left(\lambda^* e^{\phi t} \pi_t\right)$$

- The consumption is always greater than the standard of living, by an amount that is determined by matching the marginal utility to the (scaled) state price density.

2.9 Equilibrium

- We finally turn to the equilibrium.
- Unfortunately, solving for equilibrium in this case is more difficult, because changes in consumption today affect the whole consumption in the future.
- Therefore, one has to introduce the notion of "shocks" to Brownian paths, which are well described by the tools of Malliavin Calculus.
- We won't go into it (if you are interested, read the paper).
- I will only briefly describe (heuristically) the result.
- **Results (some of):**
  1. The equilibrium state price density is given by
  $$\pi_t^e \lambda = u_c\left(e_t, z_t^e\right) + \delta E_t\left[\int_t^T e^{-(\phi+a)(\tau-t)} u_v\left(e_\tau, z_\tau^e\right) d\tau\right]$$
  2. The equilibrium interest rate is (no kidding!)
  $$\begin{aligned} r_t &= \phi - \left(\pi_t^e\right)^{-1}\left\{u_{cc}(t)\,\mu_{e,t} + \frac{1}{2}u_{ccc}(t)\,\sigma_e^2\right. \\ &\quad + u_{cv}(t)\left(\delta e_t - \alpha z_t^e\right) \\ &\quad \left. + \,\delta\left[(\alpha+\phi)\,E_t\left[\int_t^T e^{-(\phi+\alpha)(\tau-t)} u_v(\tau)\,d\tau\right] - u_v(t)\right]\right\} \end{aligned}$$
  where $u(t) = u\left(e_t, z_t^e\right)$ and so is for the other derivatives.

3. The expected returns on the stock i can be rewritten as
$$E\left[\frac{dS_t^i}{S_t^i}\right]-r_t=\Gamma_t^H Cov_t\left(\frac{dS_t^i}{S_t^i},de_t\right)-Cov_t\left(\frac{dS_t^i}{S_t^i},\delta Y\left(t\right)dW_t\right)$$
where
$$\begin{aligned}\Gamma_t^H &= -\left(\pi_t^e\right)^{-1}\Big\{u_{cc}\left(t\right)+\delta\Big[\int_t^T e^{-(\phi+\alpha)(\tau-t)} \\ &\quad +\left[u_{cv}\left(t\right)+\delta u_{vv}\left(t\right)\alpha^{-1}\left(1-e^{-\alpha(t-\tau)}\right)\right]d\tau\Big]\Big\}\end{aligned}$$
and $Y\left(t\right)$ is a process that depends on the time-varying $\mu_{e,t}$ and $\sigma_{e,t}$.

- These are definitely messy formulas.
- However, they have some intuition.
- Consider for example the interest rate. The first line is standard and it has to do with risk aversion (or intertemporal elasticity) and the prudence coefficient.
- We must add a second term to take into account the fact that current consumption choices will affect future consumption.
- In fact, the agent is no longer so keen in substituting future consumption for current consumption (that is, it has an additional preference to push consumption to tomorrow). Hence, the demand for bonds increases and the interest rate decreases!
- The cross section of returns also is rather intuitive. Consider first the case where $\mu_{e,t}$ and $\sigma_{e,t}$ are constants.
- In this case (it can be shown that) $Y\left(t\right)=0$ and we have the standard C-CAPM
$$E\left[\frac{dS_t^i}{S_t^i}\right]-r_t=e_t\Gamma_t^H Cov_t\left(\frac{dS_t^i}{S_t^i},\frac{de_t}{e_t}\right)$$

- The risk premium is still given by the covariance of stock with current endowment. This is because shocks to current endowment determine the "future disutility" for given parameters μ_e and σ_e. As a consequence, the usual hedging argument goes through.
- What changes is the coefficient in front of the covariance term.
- Compared to the standard iso-elastic case, $e_t \Gamma_t^H$ is much bigger than the coefficient of risk aversion $\gamma_t = -cu_{cc}(t)/u_c(t)$, because the state-price density is moving due to the disutilities of future consumption.
- In addition, it is pointed out that even if the coefficient of relative risk aversion γ_t is constant, the coefficient $e_t \Gamma_t^H$ is still stochastic. Hence, one cannot reject constant relative risk aversion by the finding of different values across time.
- When $\mu_{e,t}$ and $\sigma_{e,t}$ are not constants, then effectively the CCAPM has a second term (difficult to quantify) that is due to the covariance between returns and the shifts in the endowment process.
- Since such shifts in the endowment process parameters also affect the future disutilities, assets that correlate with those may have trade at a premium (to hedge against those).

3 Recursive Utility

- Before discussing the continuous time version of recursive utility, called stochastic differential utility, it is useful to discuss the discrete time counterpart.

- Consider first the iso-elastic utility function

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}$$

- If C is stochastic, then $\gamma = -CU_{cc}/U_c$ is the coefficient of relative risk aversion.

- In an intertemporal model, with deterministic consumption $C_1, C_2, \ldots$ $\psi = 1/\gamma$ instead measures also the elasticity of intertemporal substitution.

- That is, the derivative of planned log consumption growth with respect to log interest rate

$$\psi = \frac{d\left(C_{t+1}/C_t\right)/\left(C_{t+1}/C_t\right)}{dR/R}$$

- This measures the willingness to exchange consumption today for consumption tomorrow, given the interest rate R.

- There is no need to have such a tight relationship between the relative risk aversion coefficient and the elasticity of intertemporal substitutions.

- They are such different concepts (one applies to stochastic variables, the other to deterministic consumption paths), that it is important to keep them separated.

- This separation is accomplished by the use of recursive utility functions.
- For example, consider a simple two period model. At time $t = 0$ you know that your consumption is C_0.
- However, at $t = 1$, you may receive the stochastic consumption $\widetilde{C}_1$.
- Given the distribution of $\widetilde{C}_1$, you can think what is the level of certain consumption at time $t = 1$ that indeed is equivalent to $\widetilde{C}_1$.
- Say this is $\overline{C}_1 = m\left(\widetilde{C}_1\right)$. Clearly, the function $m(.)$ measures the "risk-aversion."
- Now, we can compare the consumption today C_0 and the deterministic consumption tomorrow $\overline{C}_1$ by using some conventional utility function defined on two commodities $W\left(C_1, \overline{C}_2\right)$.
- Clearly, the function $W\left(C_1, \overline{C}_2\right)$ measures only the substitution preferences across the two periods and not the "risk aversion" component.
- The recursive utility functions generalize the above.
- They are in fact defined by the following ingredients:
  1. V_t is the "utility" at time t. $\widetilde{V}_{t+1}$ denotes the fact that it is stochastic in the future (as of time t or before).
  2. A certainty equivalent function $m(.|\mathcal{F}_t)$ defined on the future stochastic utility $\widetilde{V}_{t+1}$
  3. An aggregator function $W(.,.)$ defined on current consumption and the certainty equivalent function.

- Specifically, we have that the utility at time t is given by

$$V_t = W\left(C_t, m\left[\widetilde{V}_{t+1}|\mathcal{F}_t\right]\right)$$

- This preferences make it possible to separate risk aversion from intertemporal substitution: As mentioned, the certainty equivalent $m\left[\widetilde{V}_{t+1}|\mathcal{F}_t\right]$ "records" the risk aversion component, while the function $W(x, y)$ records the relative preference for a good x today or the "certainty equivalent" of utility $\widetilde{V}_{t+1}$, y, tomorrow.

- A typical parametric example for the aggregator function is the CES function

$$W(x, y) = \left((1-\delta)x^{\rho} + \delta y^{\rho}\right)^{\frac{1}{\rho}}$$

- On non-stochastic consumption, the elasticity of intertemporal substitution is $\psi = 1/(1-\rho)$. Instead, δ is the intertemporal discount factor.

- Notice that the function $W(x, y)$ is homogeneous in x and y, that is

$$\begin{aligned} W(ax, ay) &= \left((1-\delta)a^{\rho}x^{\rho} + \delta a^{\rho}y^{\rho}\right)^{\frac{1}{\rho}} \\ &= aW(x, y) \end{aligned}$$

- The standard form of Epstein-Zin-Weil recursive utility function is obtained by assuming a constant relative risk aversion utility specification for the certainty equivalent.

- That is, given a random variable $\tilde{x}$, assume

$$m(\tilde{x}) = E\left[\tilde{x}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

- Putting things together and renaming variables, one obtains

$$V_t = \left\{(1-\delta)\,C_t^{1-1/\psi} + \delta\left(E_t\left[\widetilde{V}_{t+1}^{1-\gamma}\right]\right)^{\frac{1-1/\psi}{1-\gamma}}\right\}^{\frac{1}{1-1/\psi}} \tag{22}$$

- The parameter γ can be viewed as the standard constant relative risk aversion parameter.
- Notice that if $\gamma = 1/\psi$, we obtain

$$V_t^{1-\gamma} = \left\{(1-\delta)\,C_t^{1-\gamma} + \delta\left(E_t\left[\widetilde{V}_{t+1}^{1-\gamma}\right]\right)\right\}$$

- Solving forward, one readily obtains the standard time-separable formulation

$$V_t = \left\{(1-\delta)\sum_{j=0}^{\infty}\delta^j C_{t+j}^{(1-\gamma)}\right\}^{\frac{1}{1-\gamma}}$$

- In this case, there is indifference on the timing of the resolution of uncertainty.
- In the general case (22), we instead have that $\gamma > 1/\psi$ implies a preference for early resolution of uncertainty (see Epstein-Zin (1989)).

3.1 Bellman Equation and Euler Equations

- Consider a representative agent endowed with an initial stocks of the consumption good, w_0, which can either be consumed or allocated to assets.
- Let $\mathbf{R}_{t+1} = \left(R^1_{t+1}, .., R^n_{t+1}\right)$ be the gross return on n assets between t and $t+1$.
- Assume that the risk-free asset is included in the vector $\mathbf{R}_{t+1}$.
- Let also $\boldsymbol{\vartheta}_t$ be the *fraction* of wealth invested in the n assets and I_t the value of any state-variable we may want to include in the information set.
- We then have the wealth evolution process

$$W_{t+1} = (W_t - C_t) \cdot \boldsymbol{\vartheta}_t \cdot \mathbf{R}_{t+1} \tag{23}$$

- The Bellman Equation takes the form

$$\begin{aligned} J\left(W_t, I_t\right) &= \max_{C_t, \boldsymbol{\vartheta}_t} \left\{ (1-\delta) C_t^{\rho} \right. && (24) \\ & \left. + \delta \left(E_t \left[J\left(\widetilde{W}_{t+1}, \tilde{I}_{t+1}\right)^{\alpha} \right] \right)^{\frac{\rho}{\alpha}} \right\}^{\frac{1}{\rho}} && (25) \end{aligned}$$

- where for convenience I set $\rho = 1 - 1/\psi$ and $\alpha = (1-\gamma)$.
- Notice that the homogeneity of the aggregator $W(.,.)$ and the linearity of the budget constraint implies (the conjecture) that $J(W, I)$ is also homogeneous in wealth (that is, the problem is *scale invariant*), implying the form

$$J\left(W_t, I_t\right) = \phi\left(I_t\right) W_t = \phi_t W_t \tag{26}$$

- Substituting (26) and the budget constraint (23) into (24) we obtain

$$J(W_t, I_t) = \max_{C_t, \boldsymbol{\vartheta}_t} \{(1-\delta) C_t^{\rho} \tag{27}$$

$$+\delta \left(E_t\left[\phi_{t+1}^{\alpha}\left(R_t^M\right)^{\alpha}\right]\right)^{\frac{\rho}{\alpha}} (W_t - C_t)^{\rho}\Big\}^{\frac{1}{\rho}} \tag{28}$$

$$= \max_{C_t, \boldsymbol{\vartheta}_t} \{(1-\delta) C_t^{\rho} + \delta \mu_t^{*\rho} (W_t - C_t)^{\rho}\}^{\frac{1}{\rho}} \tag{29}$$

- where $\mu_t^* = E_t\left[\phi_{t+1}^{\alpha}\left(R_{t+1}^M\right)^{\alpha}\right]^{\frac{1}{\alpha}}$ and where $R_{t+1}^M = \boldsymbol{\vartheta}_t \cdot \mathbf{R}_{t+1}$ is the return on the market portfolio.
- Maximizing over C_t yields the FOC

$$C_t^{\rho-1} = \frac{\delta}{1-\delta} (W_t - C_t)^{\rho-1} \mu_t^{*\rho}$$

- Also, the consumption is proportional to wealth (given the homogeneity again) implying

$$C_t = \psi(I_t) W_t = \psi_t W_t$$

- This implies

$$\psi_t^{\rho-1} = \frac{\delta}{1-\delta} (1-\psi_t)^{\rho-1} \mu_t^{*\rho} \tag{30}$$

- or

$$\mu_t^{*\rho} = \frac{1-\delta}{\delta}\left(\frac{\psi_t}{1-\psi_t}\right)^{\rho-1}$$

- Substituting into the value function, we obtain

$$J\left(W_t, I_t\right) = W_t\left\{(1-\delta)^{\frac{1}{\rho}}\,\psi_t^{\frac{\rho-1}{\rho}}\right\} \tag{31}$$

- Hence

$$\phi_t = (1-\delta)^{\frac{1}{\rho}}\left(\frac{C_t}{W_t}\right)^{\frac{\rho-1}{\rho}}$$

- Substituting this into (30) one obtains

$$\psi_t^{\rho-1} = \frac{\delta}{1-\delta}\left(1-\psi_t\right)^{\rho-1} E_t\left[\phi_{t+1}^{\alpha}\left(R_{t+1}^M\right)^{\alpha}\right]^{\frac{\rho}{\alpha}}$$

- or

$$E_t\left[\left(\delta\left(\frac{C_{t+1}}{C_t}\right)^{\rho-1} R_{t+1}^M\right)^{\frac{\alpha}{\rho}}\right]^{\frac{\rho}{\alpha}} = 1$$

- This is the Euler equation with respect to consumption.
- We now need to maximize with respect to $\boldsymbol{\vartheta}_t$ as well the expression in the Bellman Equation (27).
- Clearly, this maximization problem is equivalent to

$$\max_{\boldsymbol{\vartheta}_t} E_t\left[\left(\phi_{t+1}\boldsymbol{\vartheta}_t\cdot\mathbf{R}_{t+1}\right)^{\alpha}\right]^{\frac{1}{\alpha}}$$

- subject to $\boldsymbol{\vartheta}_t \cdot 1_n = 1$.
- Using the Lagrangian and taking the FOC with respect to ϑ_t^i for all i and managing the equations, one obtains the necessary conditions

$$E_t\left[\phi_{t+1}^{\alpha}\left(R_{t+1}^{M}\right)^{\alpha-1}\left(R_{t+1}^{i}-R_{t+1}^{1}\right)\right]=0 \qquad (32)$$

- Now, by our earlier finding

$$\begin{aligned}\phi_{t+1} &= (1-\delta)^{\frac{1}{\rho}}\left(\frac{C_{t+1}}{W_{t+1}}\right)^{\frac{\rho-1}{\rho}} \\ &= (1-\delta)^{\frac{1}{\rho}}\left(\frac{C_{t+1}}{W_t\left(1-\psi_t\right)R_{t+1}^{M}}\right)^{\frac{\rho-1}{\rho}} \\ &= \frac{(1-\delta)^{\frac{1}{\rho}}}{W_t\left(1-\psi_t\right)^{\frac{\rho-1}{\rho}}}\left(\frac{C_{t+1}}{R_{t+1}^{M}}\right)^{\frac{\rho-1}{\rho}}\end{aligned}$$

- and substituting all into (32), we find

$$E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\alpha}{\rho}(\rho-1)}\left(R_{t+1}^{M}\right)^{\frac{\alpha}{\rho}-1}\left(R_{t+1}^{i}-R_{t+1}^{1}\right)\right]=0 \qquad (33)$$

- In a more conventional way, we can also rewrite this as

$$E_t\left[\delta^{\frac{\alpha}{\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\alpha}{\rho}(\rho-1)}\left(R_{t+1}^{M}\right)^{\frac{\alpha}{\rho}-1}R_{t+1}^{i}\right]=1 \qquad (34)$$

- These are the Euler equations in the case of recursive utility.

- Notice that the case $\gamma = 1/\psi$ implies $\alpha = \rho$, which reduces the Euler equation to

$$E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\rho-1} R^i_{t+1}\right] = 1 \tag{35}$$

3.1.1 Jointly Log-Normal Returns

- We can use the Euler equations to obtain implications for stock returns.
- When consumption growth and returns are jointly log normal, we can solve for excess returns.
- First, from the Euler equation the risk-free rate must satisfy

$$R^f_{t+1} = \frac{1}{E_t\left[\delta^{\frac{\alpha}{\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\alpha}{\rho}(\rho-1)}(R_{t+1})^{\frac{\alpha}{\rho}-1}\right]}$$

- Assuming that consumption growth and the market returns are jointly log normal, we obtain

$$\begin{aligned} r^f_t &= \log\left(R^f_t\right) = \phi + \frac{1}{2}\left(\frac{\alpha}{\rho}-1\right)\sigma^2_m - \frac{1}{2}\frac{\alpha}{\rho\psi^2}\sigma^2_c + \frac{1}{\psi}E\left[d\log\left(C_{t+1}\right)\right] \\ &= \phi + \frac{1}{2}\frac{\alpha}{\rho}\left(\sigma^2_m - \frac{1}{\psi^2}\sigma^2_c\right) - \frac{1}{2}\sigma^2_m + \frac{1}{\psi}E\left[d\log\left(C_{t+1}\right)\right] \end{aligned}$$

- We discover that the risk free rate depends on the discount rate ϕ, on the elasticity of intertemporal substitution ψ and the coefficient of risk aversion γ.
- An increase in the elasticity of substitution ψ increases $\rho = 1 - 1/\psi$ and hence tend to decrease the equilibrium interest rate (but it is not necessarily true).

- Intuitively, if one is really inelastic about consuming today or tomorrow, he/she must be induced to exchange consumption by a large interest rate.

- Similarly, an increase in the coefficient of risk aversion γ decreases $\alpha = (1 - \gamma)$ and hence the interest rate r_t^f. Intuitively, again, a higher risk aversion increases the demand for bonds and hence lowers the interest rate.

- When $\gamma = 1/\psi$ we have the opposite result: An increase in risk-aversion increases the interest rate because it decreases the elasticity of intertemporal substitution.

- Similarly, one also obtains that the risk-premia on the various assets are

$$E_t\left[r_{t+1}^i\right] - r_t^f - \frac{1}{2}\sigma_i^2 = \frac{\alpha}{\rho\psi}\sigma_{ic} + \left(1 - \frac{\alpha}{\rho}\right)\sigma_{im} \qquad (36)$$

- We obtain a mixture of the C-CAPM and the CAPM, where the equity risk premium depends both on the covariance with consumption and the covariance with the market itself.

- An increase in risk aversion γ decreases $\alpha = 1 - \gamma$ and has the effect of decreasing the premium due to the covariance between asset i and consumption and increasing the effect coming from the covariance between asset i and the market.

- The same for an increase in the elasticity of intertemporal substitution.

- The presence of both covariance terms is clearly an effect stemming from the fact that the stochastic discount factor now depends on both the return on the market and the consumption growth.

3.1.2 Solving for an Equilibrium Model: Cecchetti-Lam-Mark (1990 -1992) Set-up

- We now use the model to solve for equilibrium in a standard Lucas (1978) set-up.
- The CLM model is an extension of Mehra-Prescott model where there are a risky asset in positive net supply and a riskless asset in zero net supply.
- Although CLM is set for the case $\gamma = 1/\psi$, it is easy to extend the results for the general Epstein-Zin-Weil preferences. (See e.g. Hung 1994)
- The risky asset (tree) pays dividends according to the process

$$D_{t+1} = D_t e^{g_t - \frac{1}{2}\sigma^2 + \sigma \varepsilon_{t+1}}$$

- It is assumed that g_t follows a $n-$state Markov chain with states g^i, $i = 1, .., n$ and transition probabilities λ_{ij}.
- Notice that in Mehra - Prescott original paper we have $\sigma = 0$.
- From above, the Euler equation for the Market Portfolio is

$$E_t \left[\left\{ \delta \left(\frac{C_{t+1}}{C_t} \right)^{\rho-1} R_{t+1} \right\}^{\frac{\alpha}{\rho}} \right] = 1 \qquad (37)$$

- where R_{t+1} is the return on the market, given by

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}$$

- Because of the scale invariance of the problem, we can conjecture that the price of the asset is given by

$$P_t = \psi\left(g_t\right) D_t$$

- For later use, define the parameter $\psi_i = \psi\left(g^i\right)$.
- We can then rewrite the Euler equation as

$$E_t\left[\left\{\delta\left(\frac{C_{t+1}}{C_t}\right)^{\rho}\frac{\left(1+\psi\left(g_{t+1}\right)\right)}{\psi\left(g_t\right)}\right\}^{\frac{\alpha}{\rho}}\right] = 1 \qquad (38)$$

- We can rewrite

$$E_t\left[e^{-\frac{\alpha}{\rho}\phi+\alpha g_t-\frac{1}{2}\alpha\sigma^2+a\sigma\varepsilon_{t+1}}\left(1+\psi\left(g_{t+1}\right)\right)^{\frac{\alpha}{\rho}}\right] = \psi\left(g_t\right)^{\frac{\alpha}{\rho}} \qquad (39)$$

- where I set $\delta = e^{-\phi}$.
- Since ε_t is assumed independent of the shocks to the g_t, we have

$$e^{-\frac{\alpha}{\rho}\phi+\alpha g_t-\frac{1}{2}\alpha(1-\alpha)\sigma^2}E_t\left[\left(1+\psi\left(g_{t+1}\right)\right)^{\frac{\alpha}{\rho}}\right] = \psi\left(g_t\right)^{\frac{\alpha}{\rho}} \qquad (40)$$

- Finally, suppose that today we hare in state i (there is perfect certainty here).

- Hence, we have

$$e^{-\frac{\alpha}{\rho}\phi+\alpha g^i-\frac{1}{2}\alpha(1-\alpha)\sigma^2}\sum_{j=1}^{n}\lambda_{ij}\left(1+\psi_j\right)^{\frac{\alpha}{\rho}}=\psi_i^{\frac{\alpha}{\rho}} \tag{41}$$

- or

$$\sum_{j=1}^{n}\lambda_{ij}\left(1+\psi_j\right)^{\frac{\alpha}{\rho}}=\psi_i^{\frac{\alpha}{\rho}}e^{-\left(-\frac{\alpha}{\rho}\phi+\alpha g^i-\frac{1}{2}\alpha(1-\alpha)\sigma^2\right)} \tag{42}$$

- This clearly defines n (non-linear) equations in n unknowns (the ψ^i's)
- In the case of CLM (and Mehra and Prescott), we have $\gamma = 1/\psi$ which implies $\alpha = \rho = 1-\gamma$
- We obtain the system of *linear* equations

$$\sum_{j=1}^{n}\lambda_{ij}\psi_j-\psi_i e^{-\left(-\frac{\alpha}{\rho}\phi+\alpha g^i-\frac{1}{2}\alpha(1-\alpha)\sigma^2\right)}=1$$

- which yields the solution

$$\boldsymbol{\psi}=\left(\boldsymbol{\Lambda}-\mathbf{D}\left(\mathbf{E}\right)\right)^{-1}\mathbf{1}_n$$

- CLM consider the case where $n=2$, estimate the model using standard ML techniques from consumptions and, given the parameters, showed by simulations that the long-term mean reversion of stocks (predictability) can simply be explained by the above process.

4 Stochastic Differential Utility

- We now "take the limit" and consider the continuous time counterpart of the approach above.
- Consider the standard set up with d Brownian motions $\mathbf{B}$ defined on a probability space $(\Omega, P, \mathcal{F})$, with the standard filtration $\{\mathcal{F}_t\}$ (generated by $\mathbf{B}_t$).
- Fix an horizon $T \in (0, \infty]$.
- The consumption processes lie in the space of square integrable processes, denoted by $\mathcal{L}^2$
- The Stochastic Differential Utility $U : \mathcal{L}^2 \rightarrow \mathcal{R}$ is defined by two primitive functions
  1. $f : \mathcal{R}_+ \times \mathcal{R} \rightarrow \mathcal{R}$;
  2. $A : \mathcal{R} \rightarrow \mathcal{R}$;
- Let a consumption process $c \in \mathcal{L}^2$ be given.
- The utility process V is the unique Ito process (if well defined) such that $V_T = 0$ and

$$dV_t = \left[-f\left(c_t, V_t\right) - \frac{1}{2} A\left(V_t\right) \sigma_{v,t} \sigma'_{v,t} \right] dt + \sigma_{v,t} dB_t \quad (43)$$

- where $\sigma_{v,t}$ is a R^d-valued, square integrable, progressively measurable process.
- We may think of the various elements as follows
  1. V_t is a "continuation utility" for the investor, given the consumption process c;

2. $f(c_t, V_t)$ is an "aggregator" similar to the one employed in the discrete time case, with the difference that it is in "differential form;"
3. $A(V_t)$ is a measure of local risk aversion, as we used in the recursive utility when we defined the "Iso-elastic" form for utility function.

- If given a consumption process c, the stochastic differential equation (43) is well defined, then the Stochastic Differential Utility U is defined as

$$U(c) = V_0$$

- the initial utility of (43).
- The pair (f, A) is called *aggregator*.
- Since $V_T = 0$ and $\int \boldsymbol{\sigma}_{v,t} d\mathbf{B}_t$ is a martingale, we can rewrite (43) as

$$V_t = E_t\left(\int_t^T \left[f(c_\tau, V_\tau) + \frac{1}{2}A(V_\tau)\,\sigma_{v,t}\sigma'_{v,t}\right] d\tau\right) \qquad (44)$$

- There are a number of properties:
  1. There are conditions under which (43) is well defined, hence a utility $U(c)$ exists;
  2. However, a closed form solution for $U(c)$ often is not available;
  3. The function $U(c)$ is monotonic and *risk averse* (see below) if $A(.) \leq 0$ and f is jointly concave and increasing in c;

4. There exists a canonical representation;
5. The Bellman equation is well defined;
6. It partially disentangle intertemporal substitution from risk aversion.
   (a) From (44) if consumption is deterministic, then $\boldsymbol{\sigma}_{v,t} = 0$ and hence intertemporal substitution must only depend on $f(c, V)$;
   (b) Risk attitudes then must then be embedded in $A(.)$. Given $f(.,.)$ and two functions $A^*(.)$, $A(.)$, let U^* and U be the utility functions corresponding to the aggregators (f, A^*) and (f, A). (Notice that the "f" is the same). If
   $$A^*(.) \leq A(.)$$
   then U^* *is more risk averse than* U, in the sense that any consumption process c rejected by U in favor of a *deterministic* process $\overline{c}$ would also be rejected by U^*. That is
   $$U(c) \leq U(\overline{c}) \Rightarrow U^*(c) \leq U^*(\overline{c})$$
   (c) Notice that $A(.)$ only establishes a comparative risk aversion for given function f. So, it does not measure the "absolute" level of risk aversion, nor can comparisons be made if f is not constant in the two aggregators.

- Some examples are in order:

- **Additive Utility**: Consider the standard time-separable case where the "continuation utility" $\overline{V}_t$ is given by the expected future utility, discounted at the subjective rate ϕ :

$$\overline{V}_t = E_t\left[\int_t^T u(c_\tau) e^{-\phi(\tau-t)} d\tau\right] \tag{45}$$

- By Ito's lemma, we have

$$d\overline{V}_t = (-u(c_t) + \phi V_t)\, dt + \sigma_v dB_t$$

- for some σ_v. Hence, we can set

$$\begin{aligned} \overline{A} &= 0 \\ \overline{f}(c,v) &= u(c) - \phi v \end{aligned}$$

- to obtain the representation (43).
- Interestingly, there are other *ordinally equivalent* representations.
- For example, consider the following

$$f(c,v) = \phi \frac{u(c) - u(v)}{u'(v)} \text{ and } A(v) = \frac{u''(v)}{u'(v)} \tag{46}$$

- It can be easily shown that the corresponding utility satisfies

$$V_t = u^{-1}(\phi \overline{V}_t) = u^{-1}\left(E_t\left[\phi \int_t^T u(c_\tau) e^{-\phi(\tau-t)} d\tau\right]\right) \tag{47}$$

- That is, U defined by (f, A) and $\overline{U}$ defined by $(\overline{f}, \overline{A})$ are ordinally equivalent, and hence represent the same preferences, in the sense that for every two consumption processes c and c' we have

$$U(c) \geq U(c') \iff \overline{U}(c) \geq \overline{U}(c')$$

- **Kreps-Porteus (Epstein-Zin-Weil) Preferences**: They correspond to

$$f(c, v) = \frac{\phi}{\rho} \frac{c^{\rho} - v^{\rho}}{v^{\rho - 1}} \text{ and } A(v) = \frac{\alpha - 1}{v} \tag{48}$$

- Closed form expressions for the corresponding utility function is not available.
- Notice that if in (46) we set $u(c) = c^{\rho}/\rho$, then we obtain that $f(c, v)$ in (48) and (46) are equal (but $A(.)$'s are not, in general!).
- Since $f(c, v)$ is what regulates intertemporal substitution, it must be the case that the elasticity of substitution is the same in the two cases, that is $\psi = (1 - \rho)^{-1}$.
- By comparing $A(.)$ in (48) and (46) we also see that they are equal if $\alpha = \rho$.
- In this case, we must obtain back the representation (47), which is ordinally equivalent to (45).
- Hence, the case $u(c) = c^{\rho}/\rho$ with $\alpha = \rho$ implies the standard time-additive utility.

- It can be shown that if $\alpha \neq \rho$, that the stochastic differential equation for the utility process V is the limit as the time interval goes to zero of the utility in discrete time, given in (22).

4.1 Ordinally Equivalent Representations

- As we saw, there are ordinally equivalent representations.
- Then, it is useful to find a way to transform a complicated problem into an easier one which yields the same solution.
- For example, in static utility maximization it is sometime useful to take a positive transformation of the original utility function to obtain an easier representation of preferences.
- We do the same here: Consider a twice-continuously differentiable function $\chi : \mathcal{R} \rightarrow \mathcal{R}$, strictly increasing with $\chi(0) = 0$.
- Two utility functions U and $\overline{U}$ are ordinally equivalent if there exists such a χ such that $\overline{U} = \chi \circ U$ (that is, for each c, we have $\overline{U}(c) = \chi(U(c))$.)
- Two aggregators (f, A) and $(\overline{f}, \overline{A})$ are ordinally equivalent if they generate ordinally equivalent utilities.
- We can see that two aggregators (f, A) and $(\overline{f}, \overline{A})$ are ordinally equivalent if there exists a change of variable χ such that

$$f(c, v) = \frac{\overline{f}(c, \chi(v))}{\chi'(v)} \text{ for all } (c, v) \in \mathcal{R}_+ \times \mathcal{R} \quad (49)$$

$$A(v) = \chi'(v)\,\overline{A}(\chi(v)) + \frac{\chi''(v)}{\chi'(v)} \quad (50)$$

- Proof: Exercise for next time!
- This gives a very nice way of solving complicated problems.
- What is bothering of the SDU utility is the term $A(v)$.
- For example, suppose I make you start with the representation

$$V_t = E_t\left(\int_t^T \left[f(c_\tau, V_\tau) + \frac{1}{2}A(V_\tau)\,\sigma_{v,\tau}\sigma'_{v,\tau}\right]d\tau\right)$$

- where

$$f(c,v) = \frac{\phi}{\rho}\frac{c^\rho - v^\rho}{v^{\rho-1}} \text{ and } A(v) = \frac{\rho-1}{v}$$

- It looks very complicated!
- However, we know that we can find a transformation such that we can obtain the ordinally equivalent representation

$$\overline{V}_t = E_t\left(\int_t^T e^{-\phi(\tau-t)}\frac{c_\tau^\rho}{\rho}d\tau\right)$$

- Well, this looks simpler, compared to the previous one!
- The important difference is that $A(.) = 0$.
- It turns out that one can do this very often.
- In fact, from (50) we see we can find χ that makes $\overline{A}(.) = 0$ by solving

$$\chi''(x) = A(x)\,\chi'(x)$$

- The solution to this is

$$\chi(v) = C_2 + C_1 \int_{v_0}^{v} e^{\int_{v_0}^{u} A(x)dx} du$$

- where v_0 is arbitrary and C_1 and C_2 are two constants such that $C_1 > 0$ and $\chi(0) = 0$.
- If $\overline{V} = \chi \circ V$ is integrable, then we must have

$$\overline{V}_t = E_t \left(\int_t^T \overline{f}(c_\tau, \overline{V}_\tau) \, d\tau \right)$$

- A clear simplification!
- In summary, given any aggregator (f, A), there exists an ordinally equivalent aggregator $(\overline{f}, 0)$, that is, such that $\overline{A}(.) = 0$. The aggregator $(\overline{f}, 0)$ or $\overline{f}$ is called "normalized aggregator."
- Notice that now both risk preferences and intertemporal substitution are "mixed up" in $\overline{f}$. Hence, by doing the transformation one gains something (simpler formula for utility) and loses something (difficult to interpret).
- The following procedure is recommended:
  1. Start with the "economic" aggregator, such as Kreps-Porteus, Epstein-Zin-Weil preferences in (48). This provides the interpretations of parameters;
  2. Find χ that makes $\overline{A}(.) = 0$ but keep in mind what the parameters mean. For example, in the Kreps-Porteus case, the normalized aggregator is (prove it!)

$$\overline{f}(c, v) = \frac{\phi}{\rho} \frac{c^\rho - (\alpha v)^{\frac{\rho}{\alpha}}}{(\alpha v)^{\left(\frac{\rho}{\alpha} - 1\right)}} \tag{51}$$

4.2 Asset Prices

- We now turn to the asset pricing implications.
- Let there be n-state variable

$$d\mathbf{X}_t = \mathbf{b}\left(\mathbf{X}_t, t\right) dt + \mathbf{a}\left(\mathbf{X}_t, t\right) d\mathbf{B}_t$$

- where $\mathbf{b} : R^n \times [0, T] \to R^n$ and $\mathbf{a} : R^n \times [0, T] \to R^{n \times d}$ and $\mathbf{B}_t$ is the d dimensional Brownian motion introduced earlier.
- Consider the value function $J(\mathbf{x}, w, t)$ denoting the maximum utility achievable in state $\mathbf{x}$, with wealth w at time t.
- Let there be N securities, following the usual Ito processes

$$d\mathbf{S}_t = \mathbf{I}_S \boldsymbol{\mu}\left(\mathbf{X}_t, t\right) dt + \mathbf{I}_S \boldsymbol{\sigma}\left(\mathbf{X}_t, t\right) d\mathbf{B}_t$$

- Let also $r(\mathbf{x}, t)$ be the interest rate in state $\mathbf{x}$. Finally. Denote the vector of excess returns by

$$\boldsymbol{\lambda}\left(\mathbf{x}, t\right) = \boldsymbol{\mu}\left(\mathbf{x}, t\right) - r\left(\mathbf{x}, t\right) \mathbf{1}_N$$

- If we denote by $\boldsymbol{\vartheta}_t$ the $(1 \times N)$ vector of fractions of wealth invested in each asset (see TN1), the wealth equation is as usual

$$dW_t = \left[W_t \boldsymbol{\vartheta}_t \boldsymbol{\lambda}\left(\mathbf{x}, t\right) + W_t r\left(\mathbf{X}_t, t\right) - c_t\right] dt + W_t \boldsymbol{\vartheta}_t \boldsymbol{\sigma}\left(\mathbf{X}_t, t\right) d\mathbf{B}_t$$

- For a *normalized aggregator*, the Bellman equation turns out to be

$$\sup_{(c,\boldsymbol{\vartheta})} \mathcal{D}^{(c,\boldsymbol{\vartheta})} J(\mathbf{x}, w, t) + f(c, J(\mathbf{x}, w, t)) = 0 \qquad (52)$$

- where

$$\mathcal{D}^{(c,\boldsymbol{\vartheta})} J(\mathbf{x}, w, t) = J_t + J_{\mathbf{x}}\mathbf{b} + J_w (w\boldsymbol{\vartheta}\boldsymbol{\lambda} + wr - c) + \frac{1}{2} tr(\boldsymbol{\Sigma})$$

- and where $tr()$ is the "trace" operator (sum of diagonal elements) and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \mathbf{a} \\ w\boldsymbol{\vartheta}\boldsymbol{\sigma} \end{pmatrix}' \begin{pmatrix} J_{xx} & J_{xw} \\ J_{wx} & J_{ww} \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ w\boldsymbol{\vartheta}\boldsymbol{\sigma} \end{pmatrix}$$

- Notice that if

$$f(c, v) = u(c) - \phi v$$

- we obtain back

$$\sup_{(c,\boldsymbol{\vartheta})} \mathcal{D}^{(c,\boldsymbol{\vartheta})} J(\mathbf{x}, w, t) + u(c) - \phi J(\mathbf{x}, w, t) = 0$$

- as in TN1.
- The simplicity of (52) is notable.
- However, it is difficult to solve the Bellman equation. It is even difficult to guess what $J(\mathbf{x}, w, t)$ looks like in this case.

- Recently, using martingale methods, Schroder and Skiadas (1999) found the optimal portfolio allocations in a number of standard simple cases.
- Here, we follow Duffie and Epstein (1992) and show that we can still obtain interesting conclusions in terms of asset pricing even without explicitly solving the Bellman equation.

4.3 A Two Factor CAPM

- Consider the FOC with respect to consumption.

$$f_c = J_w$$

- Let $c_t = C(X_t, W_t, t)$ be the optimal consumption function.
- We can then differentiate the FOC with respect to w to obtain

$$J_{ww} = f_{cc} C_w + f_{cv} J_w$$

- Differentiating the FOC with respect to $\mathbf{x}$ yields

$$J_{w\mathbf{x}} = f_{cc} C_{\mathbf{x}} + f_{cv} J_{\mathbf{x}} \tag{53}$$

- Consider now the FOC with respect to the portfolio vector $\boldsymbol{\vartheta}$.
- From the Bellman equation these are

$$J_w w \boldsymbol{\lambda} + J_{ww} \boldsymbol{\sigma}\boldsymbol{\sigma}' \boldsymbol{\vartheta}' w^2 + \boldsymbol{\sigma}\mathbf{a}' J_{\mathbf{x}w} w = 0$$

- We can obtain the excess return vector $\boldsymbol{\lambda}$ as

$$-\boldsymbol{\lambda} = \frac{f_{cc}}{J_w}\boldsymbol{\sigma}\boldsymbol{\sigma}_c' + f_{cv}\boldsymbol{\sigma}\boldsymbol{\sigma}'\boldsymbol{\vartheta}'w + \frac{f_{cv}}{J_w}\boldsymbol{\sigma}\mathbf{a}'J_{\mathbf{x}}' \qquad (54)$$

- with

$$\boldsymbol{\sigma}_c = C_{\mathbf{x}}\mathbf{a} + C_w w \boldsymbol{\vartheta}\boldsymbol{\sigma}$$

- Notice that $\boldsymbol{\sigma}_c$ is the diffusion function of consumption $C(X_t, W_t, t)$. Hence, $\boldsymbol{\Sigma}_{RC} = \boldsymbol{\sigma}\boldsymbol{\sigma}_c'$ denotes the covariance of return with consumption (assuming one representative agent).
- Also, given the optimal portfolio and the market clearing condition that $\boldsymbol{\vartheta}'W$ is the vector of values of all the securities, we have that $\boldsymbol{\sigma}'\boldsymbol{\vartheta}'W_t$ is the volatility of the market returns.
- Hence, $\boldsymbol{\Sigma}_{RM} = \boldsymbol{\sigma}\boldsymbol{\sigma}'\boldsymbol{\vartheta}'W_t$ represents the covariance with respect to the market returns.
- The third term can be interpreted in some cases. In the case of KP preferences, the homogeneity of the problem makes it possible to "guess"

$$J(x, w, t) = \psi(x, t)\, w^{\upsilon}$$

- for some υ and function ψ. Notice that in this case

$$J_w = \upsilon w^{-1} J$$
$$J_{w\mathbf{x}} = \upsilon w^{-1} J_{\mathbf{x}}$$

- Hence, we can use (53) to find

$$\frac{f_{cc}}{(\upsilon w^{-1} - f_{cv})} C_{\mathbf{x}} = J_{\mathbf{x}}$$

- We can then write the last term in (54) as

$$\begin{aligned}
\frac{f_{cv}}{J_w} \boldsymbol{\sigma} \mathbf{a}' J'_{\mathbf{x}} &= \frac{f_{cv} f_{cc}}{J_w (\upsilon w^{-1} - f_{cv})} \boldsymbol{\sigma} \mathbf{a}' C'_{\mathbf{x}} \\
&= \frac{f_{cv} f_{cc}}{J_w (\upsilon w^{-1} - f_{cv})} \boldsymbol{\sigma} \boldsymbol{\sigma}'_c + \frac{f_{cv} f_{cc} C_w}{J_w (\upsilon w^{-1} - f_{cv})} \boldsymbol{\sigma} \boldsymbol{\sigma}' \boldsymbol{\vartheta}' W_t \\
&= \frac{f_{cv} f_{cc}}{J_w (\upsilon w^{-1} - f_{cv})} \boldsymbol{\Sigma}_{RC} + \frac{f_{cv} f_{cc} C_w}{J_w (\upsilon w^{-1} - f_{cv})} \boldsymbol{\Sigma}_{MC}
\end{aligned}$$

- We can finally write

$$\boldsymbol{\lambda} = k_1 \boldsymbol{\Sigma}_{RC} + k_2 \boldsymbol{\Sigma}_{MC}$$

- This is a two factor model for the cross section of stock returns.
- It can be shown that

$$k_1 = \frac{1}{c} \frac{\alpha}{\rho \psi} \text{ and } k_2 = \frac{1}{w} \left(1 - \frac{\alpha}{\rho}\right)$$

- where $\psi = (1 - \rho)^{-1}$. This finding should be compared with (36). There is no difference, once one realizes the covariances here are with respect to the level consumption and level wealth (and not in percentage form).

4.4 An Asset Pricing Formula

- How is the state price density related to the Stochastic Differential Utility?
- We know that given a stock price process

$$d\mathbf{S}_t = \mathbf{I}_S \boldsymbol{\mu}(\mathbf{X}_t, t)\, dt + \mathbf{I}_S \boldsymbol{\sigma}(\mathbf{X}_t, t)\, d\mathbf{B}_t$$

- we can define a market price of risk process

$$\boldsymbol{\nu}(\mathbf{X}_t, t) = \boldsymbol{\sigma}(\mathbf{X}_t, t)^{-1} \boldsymbol{\lambda}(\mathbf{X}_t, t)$$

- and the state price density is just given by

$$\pi_t = e^{\left(\int_0^t -\left(r_u + \frac{1}{2}\boldsymbol{\nu}_u \boldsymbol{\nu}_u'\right)du + \int_0^t \boldsymbol{\nu}_u' d\mathbf{B}_u\right)}$$

- We saw under complete markets that we could obtain a representative agent such that

$$\pi_t = e^{-\phi t}\mathcal{U}_c(e_t)$$

- where $\mathcal{U}_c(e_t)$ is the marginal utility of the representative agent.
- Finally, given any security paying a dividend rate δ_t, we have that its price should be

$$S_t = \frac{1}{\pi_t} E_t\left[\int_t^T \pi_\tau \delta_\tau d\tau\right] = E_t\left[\int_t^T e^{-\phi(\tau - t)} \frac{\mathcal{U}_c(e_\tau)}{\mathcal{U}_c(e_t)} \delta_\tau d\tau\right]$$

- The question is whether we can find something analogous in the case of stochastic differential utility.
- It turns out that given the *normalized aggregator* f, the state-price density is given by

$$\pi_t = \exp\left(\int_0^t f_v\left(c_\tau, V_\tau\right) d\tau\right) f_c\left(c_t, V_t\right) \tag{55}$$

- Given this, one can the obtain the stock pricing formula

$$\begin{aligned} S_t &= \frac{1}{\pi_t} E_t\left[\int_t^T \pi_\tau \delta_\tau d\tau\right] \\ &= E_t\left[\int_t^T \exp\left(\int_t^\tau f_v\left(c_s, V_s\right) dt\right) \frac{f_c\left(c_\tau, V_\tau\right)}{f_c\left(c_t, V_t\right)} \delta_\tau d\tau\right] \end{aligned}$$

- Future dividends are still discounted at the marginal rate of substitution $\frac{f_c(c_\tau, V_\tau)}{f_c(c_t, V_t)}$, but they are taken to today by the exponential term $\exp\left(\int_t^\tau f_v\left(c_s, V_s\right) dt\right)$. This reflects the trade-off between consumption today and consumption tomorrow.
- Notice that in the case of time-separable preferences, we would have $f_v\left(c_s, V_s\right) = -\phi$ obtaining the usual result!

5 References

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6 Appendix

External Habit: Santos and Veronesi (2005)

Proof of pricing function: Let $\beta = 1 - \gamma$ and define

$$M_t = C_t^\beta G_t$$

we then have

$$\begin{aligned} dM_t &= +\beta M_t \frac{dC_t}{C_t} + \frac{1}{2}\beta(\beta-1)M_t\left(\frac{dC}{C}\right)^2 + C_t^\beta dG_t + \beta C_t^\beta \frac{dC}{C} dG \\ &= +\beta M_t \mu_c + \beta M_t \sigma_c dB_t^1 + \frac{1}{2}\beta(\beta-1)M_t\sigma_c^2 + C_t^\beta\left(k\left(\overline{G} - G\right)dt - \alpha(G_t - \lambda)\sigma_c dB_t\right) \\ &\quad -\alpha\beta C_t^\beta (G_t - \lambda)\sigma_c^2 \\ &= -\rho M_t + \beta M_t \mu_c + \frac{1}{2}\beta(\beta-1)M_t\sigma_c^2 + k\left(\overline{G}N_t - M_t\right)dt \\ &\quad -\alpha\beta(M_t - N_t\lambda)\sigma_c^2 + \beta M_t\sigma_c dB_t - \alpha(M_t - N_t\lambda)\sigma_c dB_t \end{aligned}$$

with

$$N_t = C_t^\beta$$

Thus

$$\begin{aligned} dN_t &= -\rho N_t + \beta N_t \frac{dC}{C} + \frac{1}{2}\beta(\beta-1)N_t\left(\frac{dC}{C}\right)^2 \\ &= -\rho N_t + \beta N_t \mu + \frac{1}{2}\beta(\beta-1)N_t\sigma_c^2 + \beta N_t \sigma_c dB_t \end{aligned}$$

Overall, we have

$$\begin{aligned} dM_t &= \left(\beta\mu_c + \frac{1}{2}\beta(\beta-1)\sigma_c^2 - k - \alpha\beta\sigma_c^2\right)M_t + \left(k\overline{G} + \alpha\beta\lambda\sigma_c^2\right)N_t + \Sigma_{M,t}dB_t^1 \\ dN_t &= \left(\beta\mu_c + \frac{1}{2}\beta(\beta-1)\sigma_c^2\right)N_t + \Sigma_{N,t}dB_t \end{aligned}$$

We can write this in a system

$$d\mathbf{Z}_t = \mathbf{A}\mathbf{Z}_t dt + \mathbf{\Sigma}_t dB_t$$

where

$$\mathbf{A} = \begin{pmatrix} \left(\beta\mu_c + \frac{1}{2}\beta(\beta-1)\sigma_c^2 - k - \alpha\beta\sigma_c^2\right) & \left(k\overline{G} + \alpha\beta\lambda\sigma_c^2\right) \\ 0 & \left(\beta\mu_c + \frac{1}{2}\beta(\beta-1)\sigma_c^2\right) \end{pmatrix}$$

As usual, we have

$$E_t[\mathbf{Z}_\tau] = \mathbf{UE}(\tau - t)\,\mathbf{U}^{-1}\mathbf{Z}_t$$

where $\mathbf{U}$ is the matrix of eigenvectors of $\mathbf{A}$, and $\mathbf{E}(\tau)$ is the diagonal matrix with $[\mathbf{E}(\tau)]_{ii} = e^{\omega_i\tau - t}$ where ω_i is the eigenvalue of $\mathbf{A}$. Thus

$$\begin{aligned} E_t[M_\tau] &= (1,0)\,\mathbf{UE}(\tau - t)\,\mathbf{U}^{-1}\mathbf{Z}_t \\ &= \sum_{i=1}^{2}\sum_{k=1}^{2} u_{1i}e^{\omega_i(\tau-t)}u_{ik}^{-1}\mathbf{Z}_{k,t} \end{aligned}$$

Apply fubini's theorem and apply this to the pricing formula to obtain

$$\begin{aligned} P_t &= C_t^\gamma S_t^\gamma \int_t^\infty E_t\left[e^{-\rho(\tau-t)}M_\tau\right]d\tau \\ &= C_t^\gamma S_t^\gamma \int_t^\infty E_t\left[e^{-\rho(\tau-t)}\sum_{i=1}^{2}\sum_{k=1}^{2} u_{1i}e^{\omega_i(\tau-t)}u_{ik}^{-1}\mathbf{Z}_{k,t}\right]d\tau \\ &= C_t^\gamma S_t^\gamma \sum_{i=1}^{2}\sum_{k=1}^{2} u_{1i}\left(\int_t^\infty E_t\left[e^{(\omega_i-\rho)(\tau-t)}\right]d\tau\right)u_{ik}^{-1}\mathbf{Z}_{k,t} \\ &= C_t^\gamma S_t^\gamma \sum_{k=1}^{2}\left(\sum_{i=1}^{2}\frac{u_{1i}u_{ik}^{-1}}{\rho-\omega_i}\right)\mathbf{Z}_{k,t} \\ &= C_t^\gamma S_t^\gamma\left(b_1C_t^{1-\gamma}G_t + b_2C_t^{1-\gamma}\right) \\ &= C_t\left(b_1 + b_2S_t^\gamma\right) \end{aligned}$$

where

$$b_k = \left(\sum_{i=1}^{2}\frac{u_{1i}u_{ik}^{-1}}{\rho-\omega_i}\right)$$

or

$$\mathbf{b} = (1,0)\,\mathbf{U}(\rho\mathbf{I} - \mathbf{W})^{-1}\,\mathbf{U}^{-1}$$

where $\mathbf{W}$ is the diagonal matrix with the eigenvalues of $\mathbf{A}$ on its principal diagonal. However, note that we also have the equality

$$(\rho\mathbf{I} - \mathbf{A})^{-1} = \mathbf{U}(\rho\mathbf{I} - \mathbf{W})^{-1}\,\mathbf{U}^{-1}$$

implying

$$\mathbf{b} = (1,0)\,(\rho\mathbf{I} - \mathbf{A})^{-1}$$

Since

$$(\rho\mathbf{I}-\mathbf{A})^{-1}=\begin{pmatrix} 1/\alpha_1 & \frac{\left(k\overline{G}+\alpha\beta\lambda\sigma_c^2\right)}{\alpha_1\alpha_2} \\ 0 & 1/\alpha_2 \end{pmatrix}$$

with

$$\begin{aligned}\alpha_1 &= \left(\rho-\beta\mu_c-\frac{1}{2}\beta\left(\beta-1\right)\sigma_c^2+k+\alpha\beta\sigma_c^2\right)\\ \alpha_2 &= \left(\rho-\beta\mu_c-\frac{1}{2}\beta\left(\beta-1\right)\sigma_c^2\right)\end{aligned}$$

Substituting $\beta=1-\gamma$, we then obtain

$$\begin{aligned}b_1 &= \frac{1}{\alpha_1}\\ b_2 &= \frac{k\overline{G}+\alpha\beta\lambda\sigma_c^2}{\alpha_1\alpha_2}\end{aligned}$$

with

$$\begin{aligned}\alpha_1 &= \rho-\left(1-\gamma\right)\mu_c+\frac{1}{2}\left(1-\gamma\right)\gamma\sigma_c^2+k+\alpha\left(1-\gamma\right)\sigma_c^2\\ \alpha_2 &= \rho-\left(1-\gamma\right)\mu_c+\frac{1}{2}\left(1-\gamma\right)\gamma\sigma_c^2\end{aligned}$$

Stochatic Differential Utility A few steps:

$$
\begin{aligned}
J\left(W_t, I_t\right) &= \left\{(1-\delta) C_t^\rho + (1-\delta)\left(\frac{\psi_t}{1-\psi_t}\right)^{\rho-1} W_t^\rho \left(1-\psi_t\right)^\rho\right\}^{\frac{1}{\rho}} && (56)\\
&= \left\{(1-\delta)\left(\psi_t^\rho W_t^\rho + \left(\frac{\psi_t}{1-\psi_t}\right)^{\rho-1} W_t^\rho \left(1-\psi_t\right)^\rho\right)\right\}^{\frac{1}{\rho}} && (57)\\
&= W_t\left\{(1-\delta)\left(\psi_t^\rho + \psi_t^{\rho-1}\left(1-\psi_t\right)\right)\right\}^{\frac{1}{\rho}} && (58)\\
&= W_t\left\{(1-\delta)^{\frac{1}{\rho}} \psi_t^{\frac{\rho-1}{\rho}}\right\} && (59)
\end{aligned}
$$

A few more steps

$$
\begin{aligned}
\psi_t^{\rho-1} &= \frac{\delta}{1-\delta}\left(1-\psi_t\right)^{\rho-1} E_t\left[\phi_{t+1}^\alpha \left(R_{t+1}^M\right)^\alpha\right]^{\frac{\rho}{\alpha}} && (60)\\
\psi_t^{\rho-1} &= \frac{\delta}{1-\delta}\left(1-\psi_t\right)^{\rho-1} E_t\left[(1-\delta)^{\frac{\alpha}{\rho}}\left(\frac{C_{t+1}}{W_{t+1}}\right)^{(\rho-1)\frac{\alpha}{\rho}}\left(R_{t+1}^M\right)^\alpha\right]^{\frac{\rho}{\alpha}} && (61)\\
\psi_t^{\rho-1} &= \delta\left(1-\psi_t\right)^{\rho-1} E_t\left[\left(\frac{C_{t+1}}{W_t\left(1-\psi_t\right) R_{t+1}^M}\right)^{(\rho-1)\frac{\alpha}{\rho}}\left(R_{t+1}^M\right)^\alpha\right]^{\frac{\rho}{\alpha}} && (62)\\
\psi_t^{\rho-1} &= \delta E_t\left[\left(\frac{C_{t+1}}{W_t}\right)^{(\rho-1)\frac{\alpha}{\rho}}\left(\frac{R_{t+1}^M}{\left(R_{t+1}^M\right)^{1-1/\rho}}\right)^\alpha\right]^{\frac{\rho}{\alpha}} && (63)\\
1 &= E_t\left[\left(\delta\left(\frac{C_{t+1}}{C_t}\right)^{\rho-1} R_{t+1}^M\right)^{\frac{\alpha}{\rho}}\right]^{\frac{\rho}{\alpha}} && (64)
\end{aligned}
$$

Proof of ordinal equivalence.

Consider the utility processes V_t and $\overline{V}_t$, each following

$$\begin{aligned} dV_t &= \left(-f\left(c_t, V_t\right) - \frac{1}{2}A\left(V_t\right)\sigma_{v,t}^2\right)dt + \sigma_{v,t}dB_t \\ d\overline{V}_t &= \left(-f\left(c_t, \overline{V}_t\right) - \frac{1}{2}A\left(\overline{V}_t\right)\sigma_{v,t}^2\right)dt + \sigma_{\overline{v},t}dB_t \end{aligned}$$

We want to find conditions on a tranformation χ makes the two utilities ordinally equivalent, that is

$$\overline{V}_t = \chi\left(V_t\right)$$

Apply Ito's lemma to find

$$\begin{aligned} d\overline{V}_t &= \chi'\left(V_t\right)dV_t + \frac{1}{2}\chi''\left(V_t\right)dV_t^2 \\ &= \left(-\chi'\left(V_t\right)f\left(c_t, V_t\right) - \frac{1}{2}\left(A\left(V_t\right)\chi'\left(V_t\right) - \chi''\left(V_t\right)\right)\sigma_{v,t}^2\right)dt \\ &\quad + \chi'\left(V_t\right)\sigma_v dB_t \\ &= \left(-\chi'\left(V_t\right)f\left(c_t, V_t\right) - \frac{1}{2}\left(\frac{A\left(V_t\right)}{\chi'\left(V_t\right)} - \frac{\chi''\left(V_t\right)}{\left[\chi'\left(V_t\right)\right]^2}\right)\left(\chi'\left(V_t\right)\sigma_{v,t}\right)^2\right)dt \\ &\quad + \chi'\left(V_t\right)\sigma_v dB_t \end{aligned}$$

Hence, comparing the last two lines with the process for $d\overline{V}_t$ above, we find that the transformation χ must satisfy

$$\begin{aligned} \chi'\left(V_t\right)f\left(c_t, V_t\right) &= \overline{f}\left(c_t, \chi\left(V_t\right)\right) \\ \left(\frac{A\left(V_t\right)}{\chi'\left(V_t\right)} - \frac{\chi''\left(V_t\right)}{\left[\chi'\left(V_t\right)\right]^2}\right) &= \overline{A}\left(\chi\left(V_t\right)\right) \end{aligned}$$

or

$$\begin{aligned} f\left(c_t, V_t\right) &= \frac{\overline{f}\left(c_t, \chi\left(V_t\right)\right)}{\chi'\left(V_t\right)} \\ A\left(V_t\right) &= \chi'\left(V_t\right)\overline{A}\left(\chi\left(V_t\right)\right) + \frac{\chi''\left(V_t\right)}{\chi'\left(V_t\right)} \end{aligned}$$

$$\begin{aligned} \chi\left(v\right) &= C_2 + C_1 \int_{v_0}^{v} e^{\int_{v_0}^{u} A(x)dx} du \\ &= C_2 + C_1 \int_{v_0}^{v} e^{\int_{v_0}^{u} (\alpha-1)x^{-1}dx} du \\ &= C_2 + C_1 \int_{v_0}^{v} e^{(\alpha-1)\log\left(\frac{u}{v_0}\right)} du \\ &= C_2 + C_1 \int_{v_0}^{v} e^{\log\left(\left(\frac{u}{v_0}\right)^{(\alpha-1)}\right)} du \\ &= C_2 + C_1 \int_{v_0}^{v} \left(\frac{u}{v_0}\right)^{(\alpha-1)} du \\ &= C_2 + C_1 \left(\left(\frac{u}{v_0}\right)^{\alpha} \frac{v_0}{\alpha}\right)_{v_0}^{v} \\ &= C_2 + C_1 \left[\left(\frac{v}{v_0}\right)^{\alpha} \frac{v_0}{\alpha} - \frac{v_0}{\alpha}\right] \end{aligned}$$

$$\begin{aligned} \chi\left(0\right) &= 0 = C_2 - C_1\left[\frac{v_0}{\alpha}\right] \\ C_2 &= C_1\left[\frac{v_0}{\alpha}\right] \end{aligned}$$

Hence

$$\chi\left(v\right) = C_2 + C_1\left(\frac{v}{v_0}\right)^{\alpha}\frac{v_0}{\alpha} - \frac{v_0}{\alpha}$$

$$
\begin{aligned}
&= C_1 \left(\frac{v}{v_0}\right)^{\alpha} \frac{v_0}{\alpha} \\
&= \frac{v^{\alpha}}{\alpha}
\end{aligned}
$$

Hence

$$\chi'(v) = v^{\alpha-1}$$

Notice then we have

$$v = (\alpha\chi(v))^{\frac{1}{\alpha}}$$

So the aggregator must satisfy

$$f(c,v) = \frac{\overline{f}(c,\chi(v))}{\chi'(v)}$$

where

$$f(c,v) = \frac{\phi}{\rho}\frac{c^{\rho} - v^{\rho}}{v^{\rho-1}}$$

Hence, we want to find $\overline{f}$ such that

$$
\begin{aligned}
\overline{f}(c,\chi(v)) &= \frac{\phi}{\rho}\frac{c^{\rho} - v^{\rho}}{v^{\rho-1}}\chi'(v) \\
&= \frac{\phi}{\rho}\frac{c^{\rho} - v^{\rho}}{v^{\rho-1}}\left[v^{\alpha-1}\right] \\
&= \frac{\phi}{\rho}\frac{c^{\rho} - [\alpha\chi(v)]^{\frac{\rho}{\alpha}}}{[\alpha\chi(v)]^{\frac{\rho-1}{\alpha}}}\left[[\alpha\chi(v)]^{\frac{\alpha-1}{\alpha}}\right] \\
&= \frac{\phi}{\rho}\frac{c^{\rho} - [\alpha\overline{v}]^{\frac{\rho}{\alpha}}}{[\alpha\overline{v}]^{\frac{\rho-1}{\alpha}}}\left[[\alpha\overline{v}]^{\frac{\alpha-1}{\alpha}}\right]
\end{aligned}
$$

$$= \frac{\phi}{\rho} \frac{c^{\rho} - [\alpha \overline{v}]^{\frac{\rho}{\alpha}}}{[\alpha \overline{v}]^{\frac{\rho}{\alpha}-1}}$$

where $\overline{v} = \chi(v)$. Hence

$$\overline{f}(c, v) = \frac{\phi}{\rho} \frac{c^{\rho} - (\alpha v)^{\frac{\rho}{\alpha}}}{(\alpha v)^{\left(\frac{\rho}{\alpha}-1\right)}}$$