

Modern Dynamic Asset Pricing Models

Lecture Notes 2.

Equilibrium with Complete Markets¹

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¹These teaching notes draw heavily on Duffie (1996, Chapters 9 and 10) and Karatzas and Shevire (1999, Chapter 3 and 4) . They are intended for students of Business 35907 only. Please, do not distribute without my prior consent.

Competitive Equilibrium

- We now use the results in TN 1 to determine the competitive equilibrium.
- The notion of equilibrium in this set up is as follows:
 1. There are m agents in the economy, each endowed with a stream of consumption good;
 2. The consumption good is immediately perishable, so that it must be consumed immediately;
 3. Agents can trade their endowments, by selling/buying financial securities;
 4. All financial securities are in zero-net supply: For every buyer there must be a seller.
- This is the standard, general equilibrium notion of a pure-exchange economy.
- Notice in particular that there is no production.

Primitives

- Let $\mathbf{B} = (B^1, \dots, B^d)$ be a d -dimensional Brownian motion defined on a complete probability space (Ω, \mathcal{F}, P) and let $\{\mathcal{F}_t\}$ denote the standard filtration of \mathbf{B} .
- Let us fix a horizon T and let the consumption space be the set L of adapted processes such that $E \left(\int_0^T c_t^2 dt \right) < \infty$.
- Suppose there are m agents, indexed by $i = 1, \dots, m$.
- Each agent receives an endowment $\{e_t^i\} \in L_+$
- Each agent has a strictly increasing utility function $U_i : L_+ \longrightarrow \mathcal{R}$.
- All agents have a common discount rate ϕ . We shall assume it constant, but it could also be a function of time.

- Each agent will maximize

$$U(c^i) = E_0 \left[\int_0^T e^{-\phi t} u_i(c_t^i) dt \right]$$

- Notice that we assume here no utility from final wealth, although it could have been inserted without any trouble.
- We shall assume everywhere that condition A in TN1 hold.

Financial Markets

- We assume complete markets. With no loss of generality, let there be d – risky securities with price processes

$$d\mathbf{S}_t = \mathbf{I}_S \boldsymbol{\mu}_t dt + \mathbf{I}_S \boldsymbol{\sigma}_t d\mathbf{B}_t$$

– where \mathbf{I}_S is the diagonal matrix with S_i on the ii –th element, and $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ are adapted processes in \mathcal{L} and \mathcal{L}^2 .

- Market completeness is achieved by assuming that $\boldsymbol{\sigma}_t$ is invertible almost everywhere.
- Also, there is a risk-free security, with short rate process r_t and price

$$\beta_t = e^{\int_0^t r_u du}$$

- As in the previous notes, define the market price of risk process

$$\boldsymbol{\nu}_t = \boldsymbol{\sigma}_t^{-1} (\boldsymbol{\mu}_t - r_t \mathbf{1}_d)$$

- Assume that the Novikov's condition is satisfied

$$E \left(\exp \left(\frac{1}{2} \int_0^T \boldsymbol{\nu}'_t \boldsymbol{\nu}_t dt \right) \right) < \infty$$

- Let

$$\xi_t = \exp \left(- \int_0^t \boldsymbol{\nu}'_t d\mathbf{B}_u - \frac{1}{2} \int_0^t \boldsymbol{\nu}'_u \boldsymbol{\nu}_u du \right)$$

- We recall that from Novikov's theorem, ξ_t is a P -martingale.
- Finally, define the state-price density process

$$\pi_t = \beta_t^{-1} \xi_t \tag{1}$$

- We saw already that π_t is such that $\mathbf{S}^\pi = \mathbf{S}\pi$ is a martingale.
- As usual, a trading strategy $(\theta^0, \boldsymbol{\theta})$ is a vector process in $\mathcal{H}^2(S)$, that is, a space with sufficient integrability conditions to rule out doubling strategies.
- For convenience, let $\tilde{\boldsymbol{\theta}} = (\theta^0, \boldsymbol{\theta})$, $\tilde{\mathbf{S}} = (\beta, \mathbf{S})$.
- A trading strategy $\tilde{\boldsymbol{\theta}} = (\theta^0, \boldsymbol{\theta})$ finances a consumption process c given income e if

$$\tilde{\boldsymbol{\theta}}_t \cdot \tilde{\mathbf{S}}_t = \int_0^t \tilde{\boldsymbol{\theta}}_u \cdot d\tilde{\mathbf{S}}_u + \int_0^t (e_u - c_u) du \geq 0 \tag{2}$$

$$\tilde{\boldsymbol{\theta}}_T \cdot \tilde{\mathbf{S}}_T = 0 \tag{3}$$

- Notice that the wealth $W_t = \tilde{\theta}_t \cdot \tilde{S}_t$ grows also because of the additional endowment that has to be taken into account.
- The last equality means that no obligations are left at horizon.
- Each agent then faces the problem

$$\sup_{(c, \theta^0, \boldsymbol{\theta}) \in \Lambda_i} U_i(c^i) \quad (4)$$

$$- \text{ where } \Lambda_i = \left\{ (c, \theta^0, \boldsymbol{\theta}) \in L_+ \times \mathcal{H}^2(S) \text{ such that } (\theta^0, \boldsymbol{\theta}) \right. \\ \left. \text{finances } c^i \text{ given } e^i \right\}$$

- A *security-spot market equilibrium* is a collection of price processes (β, \mathbf{S}) , consumption processes $(c^i)_{i=1}^m$ and trading strategies $(\tilde{\theta}^i)_{i=1}^m$ such that given (β, \mathbf{S}) , each agent solves (4) and markets clear:

$$\sum_{i=1}^m \boldsymbol{\theta}^i = 0 \text{ and } \sum_{i=1}^m c^i - e^i = 0$$

- Notice that this is an “endowment” economy, so that the aggregate consumption is just generated by the aggregate endowment.
- In equilibrium, agents trade their own endowments by selling and buying financial securities.

The Individual Agent Optimization Problem

- We first look at the optimization problem of one given agent.
- This is assumed to be “small” in the sense of taking the price processes as given and optimize his/her intertemporal utility *given the prices*.
- As one can guess, this problem is the same as the one we solved for in TN1.
- The only difference is that now our investor is not endowed with an initial wealth w but with an endowment process e .
- However, we can use the same technique used earlier to make “static” the budget constraint (2)-(3).
- Once this is accomplished, it is intuitive that the resulting optimal strategy would be similar.
- Under the assumptions above, let Q be the equivalent martingale measure defined by

$$\xi_T = \exp \left(- \int_0^T \boldsymbol{\nu}'_t d\mathbf{B}_u - \frac{1}{2} \int_0^T \boldsymbol{\nu}'_u \boldsymbol{\nu}_u du \right)$$

- Let the discounted future endowment be denoted by:

$$w^i = E^Q \left[\int_0^T \beta_t^{-1} e_t^i dt \right]$$

- Notice that the expectation is under the Q -measure.
- We recall that for given initial wealth w^i , the *static* budget constraint that we obtained in TN1 was

$$E^Q \left[\int_0^T \beta_t^{-1} c_t^i dt \right] \leq w^i$$

- Hence, in analogy with what found in TN1, we have that the dynamic budget constraint (2)-(3) can be equivalently expressed as

$$E^Q \left[\int_0^T \beta_t^{-1} c_t^i dt \right] \leq E^Q \left[\int_0^T \beta_t^{-1} e_t^i dt \right] \quad (5)$$

- The way to prove this is to go through the same steps as in Proposition 4 in TN1 and define $c_t^* = c_t - e_t$ and let $w = 0$. It is immediate that one gets (5).
- Finally, using the same method as in proposition 4 in TN1 one obtains:
- **Corollary 1:** The static budget constraint (5) is equivalent to

$$E \left[\int_0^T \pi_t c_t^i dt \right] \leq E \left[\int_0^T \pi_t e_t^i dt \right] \quad (6)$$

— where π_t is the state price density defined in (1).

- Notice that the expectation is under the original probability measure P .

Optimal Consumption and Portfolio Weights

- From the result of TN1, we then obtain the following result.
- Let $\mathcal{I}_u^i : R \longrightarrow R$ be the inverse of the instantaneous marginal utility function u_c^i , that is, it is such that for every x we have $\mathcal{I}_u^i (u_c^i (x)) = x$.
- **Proposition 1:** Let the price process (β, \mathbf{S}) be given and assume that condition A is satisfied for agent i . Then, there exists a solution to the individual investor's problem with

$$c_t^i = \mathcal{I}_u^i (\lambda_i e^{\phi t} \pi_t)$$

— where λ_i solves

$$E \left[\int_0^T \pi_t \mathcal{I}_u^i (\lambda_i e^{\phi t} \pi_t) dt \right] = E \left[\int_0^T \pi_t e^{\phi t} dt \right] \quad (7)$$

- The relation with the result in TN1 is the following:
 1. The inverse marginal utility function: In TN 1 we had $u(c, t)$, with t included in the utility function. Hence, the relationship is

$$u_c(c, t) = e^{-\phi t} u_c(c)$$

Hence, if $x = u_c(c, t)$, its inverse is

$$\mathcal{I}_u(x, t) = \mathcal{I}_u(e^{\phi t} x)$$

This explains why we have the term " $e^{\phi t}$ " inside the inverse utility function.

Optimal Consumption and Portfolio Weights

2. We defined the function

$$w(\lambda) = E \left[\int_0^T \pi_t \mathcal{I}_u(\lambda_i \pi_t, t) dt \right]$$

and we imposed $w(\lambda) = w$. Clearly, equation (7) is the same condition.

- We finally find the weights to support the optimal consumption. This is identical to our findings in TN 1, but it is useful to review them here again.
- From the proofs in TN1, we found a few important relationships that we must recall first.
- For convenience, define the wealth at time t as

$$W_t^i = \theta_t^{0,i} \beta_t + \boldsymbol{\theta}_t^i \mathbf{S}_t$$

- and the discounted wealth as

$$\widehat{W}_t^i = \beta_t^{-1} W_t^i = \theta_t^{0,i} + \boldsymbol{\theta}_t^i \mathbf{S}_t \beta_t^{-1} = \theta_t^{0,i} + \boldsymbol{\theta}_t^i \widehat{\mathbf{S}}_t$$

– where we recall that $\widehat{\mathbf{S}}_t = \mathbf{S}_t \beta_t^{-1}$ is a martingale under Q .

Optimal Consumption and Portfolio Weights

- Hence

$$d\widehat{\mathbf{S}}_t = \mathbf{I}_{\widehat{\mathbf{S}}}\boldsymbol{\sigma}_t d\widehat{\mathbf{B}}_t$$

- where $\widehat{\mathbf{B}}_t$ is a Brownian motion under Q generated by Girsanov's theorem through the formula

$$\widehat{\mathbf{B}}_t = \mathbf{B}_t + \int_0^t \boldsymbol{\nu}_u du$$

- From the dynamic budget constraint we also have

$$\widehat{W}_t^i = \theta_t^{0,i} + \boldsymbol{\theta}_t^i \cdot \widehat{\mathbf{S}}_t \tag{8}$$

$$= \int_0^t \boldsymbol{\theta}_u^i \cdot d\widehat{\mathbf{S}}_u + \int_0^t \beta_u^{-1} (e_u^i - c_u^i) du \geq 0 \tag{9}$$

- so that

$$d\widehat{W}_t^i = \beta_t^{-1} (e_t^i - c_t^i) dt + \boldsymbol{\theta}_t^i d\widehat{\mathbf{S}}_t \tag{10}$$

$$= \beta_t^{-1} (e_t^i - c_t^i) dt + \boldsymbol{\theta}_t^i \mathbf{I}_{\widehat{\mathbf{S}}}\boldsymbol{\sigma}_t d\widehat{\mathbf{B}}_t \tag{11}$$

Optimal Consumption and Portfolio Weights

- By defining $c_t^{*,i} = c_t^i - e_t^i$ and setting $w = 0$, propositions 4 and 5 in TN 1 imply that the current (discounted) wealth is just equal to the expected discounted value of future consumption minus endowment under Q :

$$\widehat{W}_t^i = E_t^Q \left(\int_t^T \beta_u^{-1} c_u^{*,i} du \right) = E_t^Q \left(\int_t^T \beta_u^{-1} (c_u^i - e_u^i) du \right)$$

- This equality is due to the assumption of complete markets: The optimal consumption stream can be thought of as a security, and \widehat{W}_t as its price at time t .
- To review how to transform these expectations under Q into expectations under P , recall that the measure Q is defined through the Radon-Nikodym derivative $\frac{dQ}{dP} = \xi_T$ and hence that we can use the property that for any random variable Z such that $E^Q(|Z|) < \infty$ we obtain

$$E^Q(Z|\mathcal{F}_t) = \frac{E(\xi_T Z|\mathcal{F}_t)}{E(\xi_T|\mathcal{F}_t)}$$

Optimal Consumption and Portfolio Weights

- We then have the following chain of equalities

$$\begin{aligned}
 \widehat{W}_t^i &= E_t^Q \left(\int_t^T \beta_u^{-1} (c_u^i - e_u^i) du \right) = \frac{E_t \left(\xi_T \int_t^T \beta_u^{-1} (c_u^i - e_u^i) du \right)}{E_t(\xi_T)} \\
 &= \frac{E_t \left(\int_t^T \xi_T \beta_u^{-1} (c_u^i - e_u^i) du \right)}{\xi_t} = \frac{E_t \left(\int_t^T E_u(\xi_T) \beta_u^{-1} (c_u^i - e_u^i) du \right)}{\xi_t} \\
 &= \frac{E_t \left(\int_t^T \xi_u \beta_u^{-1} (c_u^i - e_u^i) du \right)}{\xi_t} = \frac{E_t \left(\int_t^T \pi_u (c_u^i - e_u^i) du \right)}{\xi_t}
 \end{aligned}$$

- For notational convenience, we can define

$$J_T^i = \int_0^T \pi_u (c_u^i - e_u^i) du$$

Optimal Consumption and Portfolio Weights

- We can rewrite the discounted wealth as

$$\widehat{W}_t^i = \frac{1}{\xi_t} E_t (J_T^i - J_t^i) = \frac{1}{\xi_t} (M_t^i - J_t^i) \quad (12)$$

- where we defined the P -martingale M_t as

$$M_t^i = E_t (J_T^i) \quad (13)$$

- From the Martingale Representation Theorem, there exists a d -valued process $\boldsymbol{\eta}^i \in (\mathcal{L}^2)^d$ such that

$$M_t^i = M_0^i + \int_0^t \boldsymbol{\eta}_u^i d\mathbf{B}_u \quad (14)$$

— where we now set $M_0^i = 0$.

- Recall now that

$$d\xi_t = -\xi_t \boldsymbol{\nu}'_t d\mathbf{B}_t$$

Optimal Consumption and Portfolio Weights

- Hence, from Ito's Lemma

$$\begin{aligned}
 d\widehat{W}_t^i &= -\frac{(M_t^i - J_t^i)}{\xi_t^2} d\xi_t + \frac{1}{\xi_t} (dM_t^i - dJ_t^i) \\
 &\quad + \frac{(M_t^i - J_t^i)}{\xi_t^3} (d\xi_t)^2 - \frac{1}{\xi_t^2} d\xi_t (dM_t^i - dJ_t^i) \\
 &= \frac{(M_t^i - J_t^i)}{\xi_t} \boldsymbol{\nu}'_t d\mathbf{B}_t + \frac{1}{\xi_t} (\boldsymbol{\eta}_t^i d\mathbf{B}_t - \pi_t (c_t^i - e_t^i) dt) \\
 &\quad + \frac{(M_t^i - J_t^i)}{\xi_t} \boldsymbol{\nu}'_t \boldsymbol{\nu}_t dt + \frac{1}{\xi_t} \boldsymbol{\nu}'_t \boldsymbol{\eta}_t^{i'} dt \\
 &= \widehat{W}_t^i \boldsymbol{\nu}'_t d\mathbf{B}_t + \frac{1}{\xi_t} \boldsymbol{\eta}_t^i d\mathbf{B}_t - \beta_t^{-1} (c_t^i - e_t^i) dt \\
 &\quad + \widehat{W}_t^i \boldsymbol{\nu}'_t \boldsymbol{\nu}_t dt + \frac{1}{\xi_t} \boldsymbol{\nu}'_t \boldsymbol{\eta}_t^{i'} dt
 \end{aligned}$$

- Hence, using again $\mathbf{B}_t = \widehat{\mathbf{B}}_t - \int_0^t \boldsymbol{\nu}_u du$ we obtain

$$d\widehat{W}_t^i = -\beta_t^{-1} (c_t^i - e_t^i) dt + \frac{1}{\xi_t} \left(\boldsymbol{\eta}_t^i + \widehat{W}_t^i \boldsymbol{\nu}'_t \right) d\widehat{\mathbf{B}}_t \quad (15)$$

Optimal Consumption and Portfolio Weights

- Comparing (10) with (15), we then must have

$$\boldsymbol{\theta}_t^i \mathbf{I}_{\widehat{S}} \widehat{\boldsymbol{\sigma}}_t = \frac{1}{\xi_t} \left(\boldsymbol{\eta}_t^i + \widehat{W}_t^i \boldsymbol{\nu}'_t \right) = \frac{\boldsymbol{\eta}_t^i}{\xi_t} + \widehat{W}_t^i \boldsymbol{\nu}'_t$$

- Multiplying both sides by β_t we finally obtain

$$\boldsymbol{\theta}_t^i \mathbf{I}_S \boldsymbol{\sigma}_t = \frac{\boldsymbol{\eta}_t^i}{\pi_t} + W_t^i \boldsymbol{\nu}'_t \quad (16)$$

— where

$$W_t^i = \frac{1}{\pi_t} E_t \left(\int_t^T \pi_u (c_u^i - e_u^i) du \right) \quad (17)$$

- Using the definition

$$W_t^i = \theta_t^{0,i} \beta_t + \boldsymbol{\theta}_t^i \cdot \mathbf{S}_t$$

- we also find the allocation in bonds

$$\theta_t^{0,i} = \beta_t^{-1} (W_t^i - \boldsymbol{\theta}_t^i \cdot \mathbf{S}_t)$$

Equilibrium and the Representative Agent

- So far we merely repeated the exercise in TN1. Now, we impose market clearing conditions and obtain the equilibrium results
- In this section we are going to skip even more of the details.
- Let the aggregate endowment be denoted as

$$e = \sum_{i=1}^m e^i$$

- From the results in the previous section, we then have the following
- **Corollary 1:** In any equilibrium, we must have

$$e_t = \sum_{i=1}^m \mathcal{I}_u^i (\lambda_i e^{\phi t} \pi_t) \quad (18)$$

- where λ_i satisfy the system of equations

$$E \left[\int_0^T \pi_t (\mathcal{I}_u^i (\lambda_i e^{\phi t} \pi_t) - e_t^i) dt \right] = 0 \quad (19)$$

Equilibrium and the Representative Agent

- The converse is also true: If there exists a vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ such that (18) and (19) are satisfied, then the market is in equilibrium. In either case, the optimal consumption is given by

$$c_t^i = \mathcal{I}_u^i (\lambda_i e^{\phi t} \pi_t) \quad (20)$$

- The only part of the proof yet to determine is that if there exists $\lambda = (\lambda_1, \dots, \lambda_m)$ such that (18) and (19) are satisfied, then the resulting market is in equilibrium.
- This is true because
 1. If the vector exists, then we know that (20) maximizes utility.
 2. Hence, (18) implies that the commodity market is cleared

$$\sum_{i=1}^m c^i = \sum_{i=1}^m e^i$$

3. From the previous proof, recall that portfolio weights were determined by the martingale

$$M_t^i = \int_0^t \boldsymbol{\eta}_u^i d\mathbf{B}_u = E_t \left[\int_0^T \pi_u (c_u^i - e_u^i) du \right]$$

* From (19) we have $\sum_{i=1}^n M_t^i = 0$ which implies $\sum_{i=1}^n \eta^i = 0$.

* Summing over (17) and then (16) we finally find

$$\sum_{i=1}^m W_t^i = 0, \quad \sum_{i=1}^m \theta_t^i = 0, \quad \sum_{i=1}^m \theta_t^{0,i} = 0$$

- Equation (18) is key in the construction of a representative agent.
 - Given the representative agent, we shall characterize the parameters of the process.

- Given a vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$, let's define the function

$$\mathcal{I}_u(x, \boldsymbol{\lambda}) = \sum_{i=1}^m \mathcal{I}_u^i(\lambda_i x)$$

- This a continuous, decreasing function for $x \in (0, \infty)$.
- Define by $\mathcal{U}_c(\cdot; \boldsymbol{\lambda})$ the inverse of $\mathcal{I}(\cdot; \boldsymbol{\lambda})$, that is, that function such that for every x , we have

$$\mathcal{U}_c(\mathcal{I}_u(x, \boldsymbol{\lambda}), \boldsymbol{\lambda}) = x \tag{21}$$

- This function is strictly decreasing on $(0, \infty)$.
- Notice that the monotonicity of the function implies

$$\mathcal{I}_u(\mathcal{U}_c(c, \boldsymbol{\lambda}), \boldsymbol{\lambda}) = c$$

- Using these definitions, we can rewrite (18) as

$$e_t = \mathcal{I}_u (e^{\phi t} \pi_t, \boldsymbol{\lambda})$$

- We can then invert this function to obtain a formula for the state-price density

$$\pi_t = e^{-\phi t} \mathcal{U}_c (e_t; \boldsymbol{\lambda}) \quad (22)$$

- The form of this state price density should look familiar.
- From corollary 1, we have
- **Corollary 2:** Under condition A, a financial market is in equilibrium if and only if its state-price density is given by π_t in (22) where $\boldsymbol{\lambda}$ satisfies the system of equations

$$E \left[\int_0^T e^{-\phi t} \mathcal{U}_c (e_t; \boldsymbol{\lambda}) (\mathcal{I}_u^i (\lambda_i \mathcal{U}_c (e_t; \boldsymbol{\lambda})) - e_t^i) dt \right] = 0 \quad (23)$$

- In addition, consumption is given by

$$c_t^i = \mathcal{I}_u^i (\lambda_i \mathcal{U}_c (e_t; \boldsymbol{\lambda})) \quad (24)$$

- Corollary 2 characterizes an equilibrium, if it exists.

The Representative Agent

- Define the function

$$\mathcal{U}(c; \boldsymbol{\lambda}) = \max_{c_1 \geq 0, \dots, c_m \geq 0: \sum_{i=1}^m c_i = c} \sum_{i=1}^m \frac{1}{\lambda_i} u^i(c_i) \quad (25)$$

- We then have

- **Proposition 1:** $\mathcal{U}(c; \boldsymbol{\lambda})$ is a strictly increasing utility function satisfying condition A. In addition

$$\frac{d\mathcal{U}(c, \boldsymbol{\lambda})}{dc} = \mathcal{U}_c(c, \boldsymbol{\lambda})$$

— as defined in (21).

- The key point in the proof is to define

$$\widehat{c}_t^i = \mathcal{I}_u^i(\lambda_i \mathcal{U}_c(c; \boldsymbol{\lambda})) \quad (26)$$

- and show that effectively, these \widehat{c}_t^i satisfy (25).

- This is easy to see. Notice that (26) implies that

$$u_c^i(\tilde{c}_t^i) = \lambda_i \mathcal{U}_c(c; \boldsymbol{\lambda}) \quad (27)$$

- Hence, consider now any other (c_1, \dots, c_m) such that $\sum_{i=1}^m c_i = c$.
- From the strict concavity of the utility functions we have

$$\begin{aligned} \sum_{i=1}^m \frac{1}{\lambda_i} u^i(c^i) &\leq \sum_{i=1}^m \frac{1}{\lambda_i} (u^i(\tilde{c}^i) + (c^i - \tilde{c}^i) u_c^i(\tilde{c}^i)) \\ &= \sum_{i=1}^m \frac{1}{\lambda_i} u^i(\tilde{c}^i) + \mathcal{U}_c(c; \boldsymbol{\lambda}) \sum_{i=1}^m (c^i - \tilde{c}^i) \\ &= \sum_{i=1}^m \frac{1}{\lambda_i} u^i(\tilde{c}^i) \end{aligned}$$

- So, we have defined a representative agent who assign constant weights $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ to the various agents 1 to m .
- Given the result in (23) and (26), the weights $\lambda_1, \dots, \lambda_m$ are chosen so that the optimal consumption of the representative agent equals the aggregate endowment.

- We finally notice the following homogeneity property of the aggregate (representative) utility function:
- For every constant α , we have

$$\begin{aligned}\mathcal{U}(c, \alpha\boldsymbol{\lambda}) &= \alpha^{-1}\mathcal{U}(c, \boldsymbol{\lambda}) \\ \mathcal{U}_c(c, \alpha\boldsymbol{\lambda}) &= \alpha^{-1}\mathcal{U}_c(c, \boldsymbol{\lambda})\end{aligned}$$

- Since the marginal utility $\mathcal{U}_c(c, \alpha\boldsymbol{\lambda})$ determines the state price density, this property is convenient to renormalize the state price density.
- **Theorem 1:** There exists a vector $\boldsymbol{\lambda}$ that satisfies the system (23). In addition, if for all i

$$\frac{-cu_{cc}(c)}{u_c(c)} \leq 1$$

this solution is “unique” in the sense that any other solution $\hat{\boldsymbol{\lambda}}$ implies the existence of a constant α such that

$$\alpha\mathcal{U}_c(e_t, \boldsymbol{\lambda}) = \mathcal{U}_c(e_t, \hat{\boldsymbol{\lambda}})$$

Characterizing the Equilibrium

- We now have all the ingredients to solve back for the parameters of the price process and the interest rate process.
- First, notice that we can renormalize the state price density so that

$$\widehat{\pi}_0 = \mathcal{U}_c(e_0; \boldsymbol{\lambda}) = 1$$

- Define the process

$$\widehat{\xi}_t = \mathcal{U}_c(e_t; \boldsymbol{\lambda})$$

- By Ito's lemma we have

$$\widehat{\xi}_t = 1 + \int_0^t \mathcal{U}_{cc}(e_u; \boldsymbol{\lambda}) de_u + \frac{1}{2} \int_0^t \mathcal{U}_{ccc}(e_u; \boldsymbol{\lambda}) (de_u)^2$$

- If we assume that the aggregate endowment evolves according to the Ito process

$$de_t = e_t \mu_{e,t} dt + e_t \boldsymbol{\sigma}_{e,t} d\mathbf{B}_t$$

Characterizing the Equilibrium

- we obtain

$$\begin{aligned} \widehat{\xi}_t = & 1 + \int_0^t \left(\mathcal{U}_{cc}(e_u; \boldsymbol{\lambda}) e_u \mu_{e,u} + \frac{1}{2} \mathcal{U}_{ccc}(e_u; \boldsymbol{\lambda}) e_u^2 \boldsymbol{\sigma}_{e,u} \boldsymbol{\sigma}'_{e,u} \right) du \\ & + \int_0^t \mathcal{U}_{cc}(e_u; \boldsymbol{\lambda}) e_u \boldsymbol{\sigma}_{e,u} d\mathbf{B}_u \end{aligned} \quad (28)$$

- Recall that from (22), we must have that the state price density is

$$\widehat{\pi}_t = e^{-\phi t} \widehat{\xi}_t$$

- On the other hand, we also have that given a system of prices (β, \mathbf{S}) , each following a Ito's process as described at the beginning of the teaching notes, we must have that the state price density is

$$\pi_t = \beta_t^{-1} \xi_t$$

- where

$$\xi_t = \exp \left(-\frac{1}{2} \int_0^t \boldsymbol{\nu}_u \boldsymbol{\nu}'_u du - \int_0^t \boldsymbol{\nu}_u d\mathbf{B}_u \right)$$

Characterizing the Equilibrium

- Notice that since $\xi_0 = 1$, ξ_t satisfies the integral equation

$$\xi_t = 1 - \int_0^t \xi_u \nu'_u d\mathbf{B}_u$$

- Define

$$\tilde{\xi}_t = \pi_t e^{\phi t} = \xi_t e^{\int_0^t (\phi - r_u) du}$$

- We then have

$$\tilde{\xi}_t = 1 + \int_0^t (\phi - r_u) \tilde{\xi}_u dt - \int_0^t \xi_u \nu'_u d\mathbf{B}_u \quad (29)$$

- By definition, in equilibrium we must have

$$\hat{\pi}_t = \pi_t$$

- or

$$\hat{\xi}_t = e^{\phi t} \hat{\pi}_t = e^{\phi t} \pi_t = \tilde{\xi}_t$$

Characterizing the Equilibrium

- Hence, equating the integral equations (28) and (29) we obtain the equalities

$$\begin{aligned}(\phi - r_t) \tilde{\xi}_t &= \mathcal{U}_{cc}(e_t; \boldsymbol{\lambda}) e_t \mu_{e,t} + \frac{1}{2} \mathcal{U}_{ccc}(e_t; \boldsymbol{\lambda}) e_t^2 \boldsymbol{\sigma}_{e,t} \boldsymbol{\sigma}'_{e,t} \\ \xi_t \boldsymbol{\nu}'_t &= -\mathcal{U}_{cc}(e_t; \boldsymbol{\lambda}) e_t \boldsymbol{\sigma}_{e,t}\end{aligned}$$

- Recall that by definition $\tilde{\xi}_t = \hat{\xi}_t = \mathcal{U}_c(e_t; \boldsymbol{\lambda})$, obtaining

$$\begin{aligned}r_t &= \phi - \frac{\mathcal{U}_{cc}(e_t; \boldsymbol{\lambda}) e_t}{\mathcal{U}_c(e_t; \boldsymbol{\lambda})} \mu_{e,t} - \frac{1}{2} \frac{\mathcal{U}_{ccc}(e_t; \boldsymbol{\lambda}) e_t^2}{\mathcal{U}_c(e_t; \boldsymbol{\lambda})} \boldsymbol{\sigma}_{e,t} \boldsymbol{\sigma}'_{e,t} \\ \boldsymbol{\nu}_t &= -\frac{\mathcal{U}_{cc}(e_t; \boldsymbol{\lambda}) e_t}{\mathcal{U}_c(e_t; \boldsymbol{\lambda})} \boldsymbol{\sigma}'_{e,t}\end{aligned}$$

- We can further define the *relative risk aversion of the representative agent* as

$$\gamma(e_t; \boldsymbol{\lambda}) = -\frac{\mathcal{U}_{cc}(e_t; \boldsymbol{\lambda}) e_t}{\mathcal{U}_c(e_t; \boldsymbol{\lambda})}$$

- and the *relative prudence coefficient of the representative agent* as

$$q(e_t; \boldsymbol{\lambda}) = -\frac{\mathcal{U}_{ccc}(e_t; \boldsymbol{\lambda}) e_t}{\mathcal{U}_{cc}(e_t; \boldsymbol{\lambda})}$$

Characterizing the Equilibrium

- We can rewrite the equations as

$$r_t = \phi + \gamma(e_t; \boldsymbol{\lambda}) \mu_{e,t} - \frac{1}{2} \gamma(e_t; \boldsymbol{\lambda}) q(e_t; \boldsymbol{\lambda}) \boldsymbol{\sigma}_{e,t} \boldsymbol{\sigma}'_{e,t}$$

$$\boldsymbol{\nu}_t = \gamma(e_t; \boldsymbol{\lambda}) \boldsymbol{\sigma}'_{e,t}$$

- Therefore, we conclude that
 1. The risk-free rate increases linearly with the discount rate ϕ and the relative risk aversion coefficient $\gamma(e_t; \boldsymbol{\lambda})$, while it decreases with the relative prudence parameter $q(e_t; \boldsymbol{\lambda})$ and with the variance of the endowment process $\boldsymbol{\sigma}_{e,t} \boldsymbol{\sigma}'_{e,t}$;
 2. The market price of risk $\boldsymbol{\nu}_t$ increases linearly with the relative risk aversion coefficient and the variability of the endowment process $\boldsymbol{\sigma}'_{e,t}$ (recall that $\boldsymbol{\nu}_t$ is a vector).
- These results immediately imply the following equilibrium concept.

The Equity Premium and the Consumption CAPM

- From the definition of $\boldsymbol{\nu}_t$ we have

$$\boldsymbol{\sigma}_t \cdot \boldsymbol{\nu}_t = \boldsymbol{\mu}_t - r_t \mathbf{1}_d$$

- We then obtain

$$\boldsymbol{\mu}_t - r_t \mathbf{1}_d = \gamma(e_t; \boldsymbol{\lambda}) \boldsymbol{\sigma}_t \boldsymbol{\sigma}'_{e,t}$$

- That is

$$E \left[\frac{dS_t^i}{S_t^i} \right] - r_t dt = \gamma(e_t; \boldsymbol{\lambda}) \times \text{Cov}_t \left(\frac{dS_t^i}{S_t^i}, \frac{de_t}{e_t} \right) \quad (30)$$

- That is, the expected excess returns of asset i depends on the relative risk aversion and the covariance between asset i and the aggregate endowment process.

- There are two puzzles here:

1. Equation (30) must hold for the market as a whole.

- The correlation between stock returns and consumption growth is anywhere between .12 and -.05 (see Campbell and Cochrane (1999), Table 7)
- Even with the optimistic assumptions of correlation = .12, volatility of return = .17 and volatility of consumption growth = .02, we have expected excess return = .4% even with $\gamma = 10$.
- So, $\gamma = 100$ would be needed here to obtain an equity premium = 4%.
- Note that the risk free rate may actually end up being reasonable if γ is high enough: For instance, $\gamma = 100$ produces $r = 0$ if $\phi = .02$, $\mu_c = .02$ and $\sigma_e = .02$.
- The reason is that the precautionary saving motive $-1/2\gamma * (\gamma + 1)\sigma_e^2$ kicks in.

2. The second puzzle is about the cross-section: The consumption CAPM does not seem to work.

- Using the Fama French size/book-to-market portfolios as test portfolios, a Fama-MacBeth regression of returns on consumption growth yields an insignificant coefficient = .22 and cross-sectional $R^2 = 16\%$ (see Lettau and Ludvigson (2001, Table 3).
- Recent papers however point at noise in the consumption data and they show that if one uses more lags and leads to compute consumption growth, the result works out.
- Typical pitfall: Note that even if a Fama-MacBeth coefficient turns out to be positive and significant (and the R^2 is high) there is still an issue of economic significance: The γ needed to rationalize the result may be too high, as in the case of the aggregate market.

The CAPM

- From (30) it is possible to find a “beta” relationship.
- Define the $1 \times d$ process ψ_t using the relationship

$$\psi_t \cdot \mathbf{I}_S \cdot \sigma_t = e_t \cdot \sigma_{e,t}$$

- We can consider ψ_t as a self-financing strategy (θ^0, θ) by choosing $\theta_t^i = \psi_t^i$ and θ_t^0 to meet the self-financing constraint.
- Let S_t^ψ be the value of the portfolio $S_t^\psi = \psi_t \cdot S_t$. We then have

$$\frac{dS_t^\psi}{S_t^\psi} = \mu_{\psi,t} dt + \sigma_{\psi,t} d\mathbf{B}_t$$

— where

$$\sigma_{\psi,t} = \frac{1}{S_t^\psi} \psi_t \cdot \mathbf{I}_S \cdot \sigma_t = \frac{e_t}{S_t^\psi} \sigma_{e,t}$$

— and $\mu_{\psi,t}$ satisfies the condition (30)

$$\mu_{\psi,t} - r_t = \gamma(e_t; \boldsymbol{\lambda}) \times \sigma_{\psi,t} \sigma_{e,t}' \tag{31}$$

The CAPM

- Hence, we finally have that for all $i = 1, \dots, n$

$$\begin{aligned}\mu_t^i - r_t &= \gamma(e_t; \boldsymbol{\lambda}) \boldsymbol{\sigma}_t^i \boldsymbol{\sigma}'_{e,t} \\ &= \gamma(e_t; \boldsymbol{\lambda}) \times \frac{S_t^\psi}{e_t} \times \boldsymbol{\sigma}_t^i \boldsymbol{\sigma}'_{\psi,t}\end{aligned}$$

- Since this holds for the portfolio ψ as well, that is

$$\mu_t^\psi - r_t = \gamma(e_t; \boldsymbol{\lambda}) \times \frac{S_t^\psi}{e_t} \times \boldsymbol{\sigma}_{\psi,t} \boldsymbol{\sigma}'_{\psi,t}$$

- by substituting for the common term $\gamma(e_t; \boldsymbol{\lambda}) \times \frac{S_t^\psi}{e_t}$ we obtain the CAPM “beta” relationship

$$E_t \left(\frac{dS_t^i}{S_t^i} \right) - r_t dt = \beta_t^i \times \left(E_t \left(\frac{dS_t^\psi}{S_t^\psi} \right) - r_t \mathbf{1}_d \right)$$

— where

$$\beta_t^i = \frac{\boldsymbol{\sigma}_t^i \cdot \boldsymbol{\sigma}'_{\psi,t}}{\boldsymbol{\sigma}_{\psi,t} \cdot \boldsymbol{\sigma}'_{\psi,t}} = \frac{\text{cov}_t \left(dS_t^i / S_t^i, dS_t^\psi / S_t^\psi \right)}{\text{var}_t \left(dS_t^\psi / S_t^\psi \right)}$$

The CAPM

- **Note:**

- The (conditional) CAPM works with respect to that asset that is perfectly correlated with the endowment process (and thus the stochastic discount factor).
- This need not be the market portfolio. The existence of labor income, for instance, generate a wedge between the stochastic discount factor and the market portfolio: Thus, the CAPM is not supposed to be working (theoretically) with respect to the market portfolio.
- But even in the case where there is no labor income it is not obvious that the market portfolio is perfectly correlated with the endowment process.
- We will do more on this later on in the course.

An Asset Pricing Relationship

- Suppose that agent i sells a financial asset whose payment is its entire endowment (or a fraction of it), that is e_t^i .
- What is the fair price of this stream of payments?
- The above results imply that the price at time t of this claim is

$$S_t^{e^i} = \frac{1}{\pi_t} E_t \left[\int_t^T \pi_u e_u^i du \right] \quad (32)$$

$$= \frac{1}{\mathcal{U}_c(e_t)} E_t \left[\int_t^T e^{-\phi(u-t)} \mathcal{U}_c(e_u) e_u^i du \right] \quad (33)$$

- In fact, if this was not true one could find a trading strategy $(\theta_0, \boldsymbol{\theta})$ that finances e_u^i and whose value is (32). This in turn generates an arbitrage opportunity.
- Indeed, the value of total endowment process is simply

$$S_t = \frac{1}{\mathcal{U}_c(e_t)} E_t \left[\int_t^T e^{-\phi(u-t)} \mathcal{U}_c(e_u) e_u du \right] \quad (34)$$

- We will use this pricing equation often.

A simple example: Log Utility

- Suppose all the agents have logarithmic utility

$$u^i(c_t) = \log(c_t)$$

- so that the inverse of the marginal utility is given by

$$\mathcal{I}_u^i(x) = x^{-1}$$

- In this case, the marginal utility of the representative investor is

$$\mathcal{U}_c(c) = \frac{1}{c} \sum_{i=1}^m \lambda_i^{-1}$$

- We can renormalize the weights so that $\sum_{i=1}^m \lambda_i^{-1} = e_0$, which we can assume positive.

- We then have

$$\mathcal{U}_c(e_0) = 1$$

- The vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ has to satisfy the system of equations (23),

$$E \left[\int_0^T e^{-\phi t} \mathcal{U}_c(e_t; \boldsymbol{\lambda}) (\mathcal{I}_u^i(\lambda_i \mathcal{U}_c(e_t; \boldsymbol{\lambda})) - e_t^i) dt \right] = 0 \quad (35)$$

A simple example: Log Utility

- which becomes

$$E \left[\int_0^T e^{-\phi t} \frac{e_0}{e_t} \left(\left(\lambda_i \frac{e_0}{e_t} \right)^{-1} - e_t^i \right) dt \right] = 0 \quad (36)$$

- and in turn

$$\lambda_i^{-1} = e_0 \frac{E \left[\int_0^T e^{-\phi t} \frac{e_t^i}{e_t} dt \right]}{\int_0^T e^{-\phi t} dt} \quad (37)$$

- Hence

$$\pi_t = e^{-\phi t} \mathcal{U}_c(e_t; \boldsymbol{\lambda}) = e^{-\phi t} \frac{e_0}{e_t}$$

- We then have that

$$c_t^i = \mathcal{I}_u(\lambda_i e^{\phi t} \pi_t) = \mathcal{I}_u \left(\lambda_i \frac{e_0}{e_t} \right) = \lambda_i^{-1} \frac{e_t}{e_0} \quad (38)$$

A simple example: Log Utility

- Finally, we have

$$\gamma(e_t; \boldsymbol{\lambda}) = -\frac{e_t \mathcal{U}_{cc}(e_t, \lambda)}{\mathcal{U}_c(e_t, \lambda)} = -\frac{e_t (-e_0/e_t^2)}{e_0/e_t} = 1$$

$$q(e_t; \boldsymbol{\lambda}) = -\frac{e_t \mathcal{U}_{ccc}(e_t, \lambda)}{\mathcal{U}_{cc}(e_t, \lambda)} = -\frac{e_t 2e_0/e_t^3}{(-e_0/e_t^2)} = 2$$

- Hence, the condition for the market equilibrium are

$$r_t = \phi + \mu_{e,t} - \boldsymbol{\sigma}_{e,t} \boldsymbol{\sigma}'_{e,t}$$

$$\boldsymbol{\nu}_t = \boldsymbol{\sigma}'_{e,t}$$

An Alternative Formula for the C-CAPM

- Before commenting further, notice also that an alternative expression for the (30) can be obtained as follows.
- We know that at an optimum, the following condition holds (see (27))

$$u_c^j(\tilde{c}_t^j) = \lambda_j \mathcal{U}_c(e; \boldsymbol{\lambda}) \quad (39)$$

- Hence, rather than using the representative agent utility function, we may think of using the marginal utility of agent j defined on the optimal consumption path \tilde{c}^j .
- Equation (39) ensures that the two approaches are identical.
- Define the process

$$\tilde{\xi}_t^j = \frac{1}{\lambda_j} u_c^j(\tilde{c}_t^j)$$

- Define the state-price density of agent j as

$$\hat{\pi}_t^j = \beta_t^{-1} \tilde{\xi}_t^j$$

An Alternative Formula for the C-CAPM

- By going through the same type of calculation, it is clear that everywhere we can substitute the “representative agent” utility and endowment, with agent i utility and *consumption*.
- As a consequence, one then obtains

$$\begin{aligned} E \left[\frac{dS_t^i}{S_t^i} \right] - r_t dt &= \gamma^j \left(\tilde{c}_t^j \right) \times Cov_t \left(\frac{dS_t^i}{S_t^i}, \frac{d\tilde{c}_t^j}{\tilde{c}_t^j} \right) \\ &= a^j \left(\tilde{c}_t^j \right) \times Cov_t \left(\frac{dS_t^i}{S_t^i}, d\tilde{c}_t^j \right) \end{aligned}$$

- where

$$a^j \left(\tilde{c}_t^j \right) = - \frac{u_{cc}^j \left(\tilde{c}_t^j \right)}{u_c^j \left(\tilde{c}_t^j \right)}$$

- is the coefficient of absolute risk aversion of agent j .
- Divide now both sides by $a^j \left(\tilde{c}_t^j \right)$ and sum across $j = 1, \dots, m$. Since from the market clearing condition $\sum_{j=1}^m \tilde{c}_t^j = e_t$ we must have

$$\sum_{j=1}^m Cov_t \left(\frac{dS_t^i}{S_t^i}, d\tilde{c}_t^j \right) = Cov_t \left(\frac{dS_t^i}{S_t^i}, de_t \right)$$

An Alternative Formula for the C-CAPM

- we obtain

$$E \left[\frac{dS_t^i}{S_t^i} \right] - r_t dt = \Gamma(\mathbf{c}_t) \times Cov_t \left(\frac{dS_t^i}{S_t^i}, de_t \right)$$

- The coefficient

$$\Gamma(\mathbf{c}_t) = \frac{1}{\sum_{j=1}^m a^j (\tilde{c}_t^j)^{-1}}$$

- is the coefficient of absolute risk aversion of the market itself.