Teaching Notes #2
Equilibrium with Complete Markets[1]

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Business 35909 Spring 2005

This Version: November 17, 2005

[1]These teaching notes draw heavily on Duffie (1996, Chapters 9 and 10) and Karatzas and Shevre (1999, Chapter 3 and 4) . They are intended for students of Business 35909 only. Please, do not distribute without my prior consent.

1 Competitive Equilibrium

- We now use the results in TN 1 to determine the competitive equilibrium.
- The notion of equilibrium in this set up is as follows:
  1. There are m agents in the economy, each endowed with a stream of consumption good;
  2. The consumption good is immediately perishable, so that it must be consumed immediately;
  3. Agents can trade their endowments, by selling/buying financial securities;
  4. All financial securities are in zero-net supply: For every buyer there must be a seller.
- This is the standard, general equilibrium notion of a pure-exchange economy.
- Notice in particular that there is no production.

1.1 Primitives

- Let $\mathbf{B} = \left(B^1, ..., B^d\right)$ be a d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t\}$ denote the standard filtration of $\mathbf{B}$.
- Let us fix a horizon T and let the consumption space be the set L of adapted processes such that $E\left(\int_0^T c_t^2 dt\right) < \infty$.
- Suppose there are m agents, indexed by $i = 1, .., m$.
- Each agent receives an endowment $\{e_t^i\} \in L_+$

- Each agent has a strictly increasing utility function $U_i : L_+ \longrightarrow \mathcal{R}$.
- All agents have a common discount rate ϕ. We shall assume it constant, but it could also be a function of time.
- Each agent will maximize

$$U\left(c^i\right) = E_0\left[\int_0^T e^{-\phi t} u_i\left(c_t^i\right) dt\right]$$

- Notice that we assume here no utility from final wealth, although it could have been inserted without any trouble.
- We shall assume everywhere that condition A in TN1 hold.

1.2 Financial Markets

- We assume complete markets. With no loss of generality, let there be $d-$ risky securities with price processes

$$d\mathbf{S}_t = \mathbf{I}_S \boldsymbol{\mu}_t dt + \mathbf{I}_S \boldsymbol{\sigma}_t d\mathbf{B}_t$$

- where $\mathbf{I}_S$ is the diagonal matrix with S_i on the $ii-$th element, and $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ are adapted processes in $\mathcal{L}$ and $\mathcal{L}^2$.
- Market completeness is achieved by assuming that $\boldsymbol{\sigma}_t$ is invertible almost everywhere.
- Also, there is a risk-free security, with short rate process r_t and price

$$\beta_t = e^{\int_0^t r_u du}$$

- As in the previous notes, define the market price of risk process

$$\boldsymbol{\nu}_t = \boldsymbol{\sigma}_t^{-1}\left(\boldsymbol{\mu}_t - r_t \mathbf{1}_d\right)$$

- Assume that the Novikov's condition

$$E\left(\exp\left(\frac{1}{2}\int_0^T \boldsymbol{\nu}_t' \boldsymbol{\nu}_t dt\right)\right) < \infty$$

- is satisfied and let

$$\xi_t = \exp\left(-\int_0^t \boldsymbol{\nu}_t' d\mathbf{B}_u - \frac{1}{2}\int_0^t \boldsymbol{\nu}_u' \boldsymbol{\nu}_u du\right)$$

- We recall that from Novikov's theorem, ξ_t is a P-martingale.
- Finally, define the state-price density process

$$\pi_t = \beta_t^{-1} \xi_t \tag{1}$$

- We saw already that π_t is such that $\mathbf{S}^\pi = \mathbf{S}\pi$ is a martingale.
- As usual, a trading strategy $\left(\theta^0, \boldsymbol{\theta}\right)$ is a vector process in $\mathcal{H}^2(S)$, that is, a space with sufficient integrability conditions to rule out doubling strategies.
- For convenience, let $\widetilde{\boldsymbol{\theta}} = \left(\theta^0, \boldsymbol{\theta}\right)$, $\widetilde{\mathbf{S}} = (\beta, \mathbf{S})$.
- A trading strategy $\widetilde{\boldsymbol{\theta}} = \left(\theta^0, \boldsymbol{\theta}\right)$ *finances a consumption process c* given income e if

$$\widetilde{\boldsymbol{\theta}}_t \cdot \widetilde{\mathbf{S}}_t = \int_0^t \widetilde{\boldsymbol{\theta}}_u \cdot d\widetilde{\mathbf{S}}_u + \int_0^t (e_u - c_u)\, du \geq 0 \tag{2}$$

$$\widetilde{\boldsymbol{\theta}}_T \cdot \widetilde{\mathbf{S}}_T = 0 \tag{3}$$

- Notice that the wealth $W_t = \widetilde{\boldsymbol{\theta}}_t \cdot \widetilde{\mathbf{S}}_t$ grows also because of the additional endowment that has to be taken into account.

- The last equality means that no obligations are left at horizon.

- Each agent then faces the problem

$$\sup_{\left(c,\theta^0,\boldsymbol{\theta}\right)\in\boldsymbol{\Lambda}_i} U_i\left(c^i\right) \tag{4}$$

- where $\boldsymbol{\Lambda}_i = \left\{ \begin{array}{c} \left(c,\theta^0,\boldsymbol{\theta}\right) \in L_+ \times \mathcal{H}^2\left(S\right) \text{ such that } \left(\theta^0,\boldsymbol{\theta}\right) \\ \text{finances } c^i \text{ given } e^i \end{array} \right\}$

- A *security-spot market equilibrium* is a collection of price processes $(\beta, \mathbf{S})$, consumption processes $\left(c^i\right)_{i=1}^m$ and trading strategies $\left(\widetilde{\theta}^i\right)_{i=1}^m$ such that given $(\beta, \mathbf{S})$, each agent solves (4) and markets clear:

$$\sum_{i=1}^m \boldsymbol{\theta}^i = 0 \text{ and } \sum_{i=1}^m c^i - e^i = 0$$

- Notice that this is an "endowment" economy, so that the aggregate consumption is just generated by the aggregate endowment.

- In equilibrium, agents trade their own endowments by selling and buying financial securities.

2 The Individual Agent Optimization Problem

- We first look at the optimization problem of one given agent.
- This is assumed to be "small" in the sense of taking the price processes as given and optimize his/her intertemporal utility *given the prices*.
- As one can guess, this problem is the same as the one we solved for in TN1.
- The only difference is that now our investor is not endowed with an initial wealth w but with an endowment process e.
- However, we can use the same technique used earlier to make "static" the budget constraint (2)-(3).
- Once this is accomplished, it is intuitive that the resulting optimal strategy would be similar.
- Under the assumptions above, let Q be the equivalent martingale measure defined by

$$\xi_T = \exp\left(-\int_0^T \boldsymbol{\nu}_t' d\mathbf{B}_u - \frac{1}{2}\int_0^T \boldsymbol{\nu}_u' \boldsymbol{\nu}_u du\right)$$

- Let the discounted future endowment be denoted by:

$$w^i = E^Q\left[\int_0^T \beta_t^{-1} e_t^i dt\right]$$

- Notice that the expectation is under the $Q-$ measure.

- We recall that for given initial wealth w^i, the *static* budget constraint that we obtained in TN1 was

$$E^Q\left[\int_0^T \beta_t^{-1} c_t^i dt\right] \leq w^i$$

- Hence, in analogy with what found in TN1, we have that the dynamic budget constraint (2)-(3) can be equivalently expressed as

$$E^Q\left[\int_0^T \beta_t^{-1} c_t^i dt\right] \leq E^Q\left[\int_0^T \beta_t^{-1} e_t^i dt\right] \qquad (5)$$

- The way to prove this is to go through the same steps as in Proposition 4 in TN1 and define $c_t^* = c_t - e_t$ and let $w = 0$. It is immediate that one gets (5).

- Finally, using the same method as in proposition 4 in TN1 one obtains:

- **Corollary 1**: The static budget constraint (5) is equivalent to

$$E\left[\int_0^T \pi_t c_t^i dt\right] \leq E\left[\int_0^T \pi_t e_t^i dt\right] \qquad (6)$$

- where π_t is the state price density defined in (1).

- Notice that the expectation is under the original probability measure P.

2.1 Optimal Consumption

- From the result of TN1, we then obtain the following result.
- Let $\mathcal{I}_u^i : R \longrightarrow R$ be the inverse of the instantaneous marginal utility function u_c^i, that is, it is such that for every x we have $\mathcal{I}_u^i \left(u_c^i \left(x\right)\right) = x$.
- **Proposition 1**: Let the price process $(\beta, \mathbf{S})$ be given and assume that condition A is satisfied for agent i. Then, there exists a solution to the individual investor's problem with

$$c_t^i = \mathcal{I}_u^i \left(\lambda_i e^{\phi t} \pi_t\right)$$

- where λ_i solves

$$E\left[\int_0^T \pi_t \mathcal{I}_u^i \left(\lambda_i e^{\phi t} \pi_t\right) dt\right] = E\left[\int_0^T \pi_t e_t^i dt\right] \quad (7)$$

- The analogy with the result in section 8.1.1 in TN1 is the following:

  1. The inverse marginal utility function: In section 8.1.1 we had $u(c,t)$, with t included in the utility function. Hence, the relationship is

  $$u_c(c,t) = e^{-\phi t} u_c(c)$$

  Hence, if $x = u_c(c,t)$, its inverse is

  $$\mathcal{I}_u(x,t) = \mathcal{I}_u\left(e^{\phi t} x\right)$$

  This explains why we have the term "$e^{\phi t}$" inside the inverse utility function.

2. We defined the function

$$w(\lambda) = E\left[\int_0^T \pi_t \mathcal{I}_u(\lambda_i \pi_t, t)\, dt\right]$$

and we imposed $w(\lambda) = w$. Clearly, equation (7) is the same condition.

2.2 Optimal Portfolio Weights

- As we pointed out earlier, with complete markets it is necessary only to find the optimal consumption.
- We can find the optimal strategy that finances consumption as a residual. We recall the method here again.
- From the proofs in TN1, we found a few important relationships that we must recall first.
- For convenience, define the wealth at time t as

$$W_t^i = \theta_t^{0,i}\beta_t + \boldsymbol{\theta}_t^i \mathbf{S}_t$$

- and the discounted wealth as

$$\widehat{W}_t^i = \beta_t^{-1} W_t^i = \theta_t^{0,i} + \boldsymbol{\theta}_t^i \mathbf{S}_t \beta_t^{-1} = \theta_t^{0,i} + \boldsymbol{\theta}_t^i \widehat{\mathbf{S}}_t$$

- where we recall that $\widehat{\mathbf{S}}_t = \mathbf{S}_t \beta_t^{-1}$ is a martingale under Q.
- Hence

$$d\widehat{\mathbf{S}}_t = \mathbf{I}_{\widehat{S}} \boldsymbol{\sigma}_t d\widehat{\mathbf{B}}_t$$

- where $\widehat{\mathbf{B}}_t$ is a Brownian motion under Q generated by Girsanov's theorem through the formula

$$\widehat{\mathbf{B}}_t = \mathbf{B}_t + \int_0^t \boldsymbol{\nu}_u du$$

- From the dynamic budget constraint we also have

$$\begin{aligned} \widehat{W}_t^i &= \theta_t^{0,i} + \boldsymbol{\theta}_t^i \cdot \widehat{\mathbf{S}}_t & (8) \\ &= \int_0^t \boldsymbol{\theta}_u^i \cdot d\widehat{\mathbf{S}}_u + \int_0^t \beta_u^{-1} \left(e_u^i - c_u^i \right) du \geq 0 & (9) \end{aligned}$$

- so that

$$\begin{aligned} d\widehat{W}_t^i &= \beta_t^{-1} \left(e_t^i - c_t^i \right) dt + \boldsymbol{\theta}_t^i d\widehat{\mathbf{S}}_t & (10) \\ &= \beta_t^{-1} \left(e_t^i - c_t^i \right) dt + \boldsymbol{\theta}_t^i \mathbf{I}_{\widehat{S}} \boldsymbol{\sigma}_t d\widehat{\mathbf{B}}_t & (11) \end{aligned}$$

- By defining $c_t^{*,i} = c_t^i - e_t^i$ and setting $w = 0$, propositions 4 and 5 in TN 1 imply that the current (discounted) wealth is just equal to the expected discounted value of future consumption minus endowment under Q:

$$\widehat{W}_t^i = E_t^Q \left(\int_t^T \beta_u^{-1} c_u^{*,i} du \right) = E_t^Q \left(\int_t^T \beta_u^{-1} \left(c_u^i - e_u^i \right) du \right)$$

- This equality is due to the assumption of complete markets: The optimal consumption stream can be thought of as a security, and $\widehat{W}_t$ as its price at time t.

- To review how to transform these expectations under Q into expectations under P, recall that the measure Q is defined through the Radon-Nikodym derivative $\frac{dQ}{dP} = \xi_T$ and hence that we can use the property that for any random variable Z such that $E^Q(|Z|) < \infty$ we obtain

$$E^Q(Z|\mathcal{F}_t) = \frac{E(\xi_T Z|\mathcal{F}_t)}{E(\xi_T|\mathcal{F}_t)}$$

- We then have the following chain of equalities

$$\begin{aligned}
\widehat{W}_t^i &= E_t^Q\left(\int_t^T \beta_u^{-1}\left(c_u^i - e_u^i\right)du\right) = \frac{E_t\left(\xi_T \int_t^T \beta_u^{-1}\left(c_u^i - e_u^i\right)du\right)}{E_t(\xi_T)} \\
&= \frac{E_t\left(\int_t^T \xi_T \beta_u^{-1}\left(c_u^i - e_u^i\right)du\right)}{\xi_t} = \frac{E_t\left(\int_t^T E_u(\xi_T)\beta_u^{-1}\left(c_u^i - e_u^i\right)du\right)}{\xi_t} \\
&= \frac{E_t\left(\int_t^T \xi_u \beta_u^{-1}\left(c_u^i - e_u^i\right)du\right)}{\xi_t} = \frac{E_t\left(\int_t^T \pi_u\left(c_u^i - e_u^i\right)du\right)}{\xi_t}
\end{aligned}$$

- For notational convenience, we can define

$$J_T^i = \int_0^T \pi_u\left(c_u^i - e_u^i\right)du$$

- so that we can rewrite the discounted wealth as

$$\widehat{W}_t^i = \frac{1}{\xi_t}E_t\left(J_T^i - J_t^i\right) = \frac{1}{\xi_t}\left(M_t^i - J_t^i\right) \tag{12}$$

- where we defined the $P-$martingale M_t as

$$M_t^i = E_t\left(J_T^i\right) \tag{13}$$

- From the Martingale Representation Theorem, there exists a $d-$valued process $\boldsymbol{\eta}^i \in \left(\mathcal{L}^2\right)^d$ such that

$$M_t^i = M_0^i + \int_0^t \boldsymbol{\eta}_u^i d\mathbf{B}_u \tag{14}$$

- where we now set $M_0^i = 0$.
- Recall now that

$$d\xi_t = -\xi_t \boldsymbol{\nu}_t' d\mathbf{B}_t$$

- Hence, from Ito's Lemma

$$\begin{aligned} d\widehat{W}_t^i &= -\frac{\left(M_t^i - J_t^i\right)}{\xi_t^2} d\xi_t + \frac{1}{\xi_t}\left(dM_t^i - dJ_t^i\right) \\ &\quad + \frac{\left(M_t^i - J_t^i\right)}{\xi_t^3}\left(d\xi_t\right)^2 - \frac{1}{\xi_t^2} d\xi_t \left(dM_t^i - dJ_t^i\right) \\ &= \frac{\left(M_t^i - J_t^i\right)}{\xi_t} \boldsymbol{\nu}_t' d\mathbf{B}_t + \frac{1}{\xi_t}\left(\boldsymbol{\eta}_t^i d\mathbf{B}_t - \pi_t \left(c_t^i - e_t^i\right) dt\right) \\ &\quad + \frac{\left(M_t^i - J_t^i\right)}{\xi_t} \boldsymbol{\nu}_t' \boldsymbol{\nu}_t dt + \frac{1}{\xi_t} \boldsymbol{\nu}_t' \boldsymbol{\eta}_t^{i\prime} dt \\ &= \widehat{W}_t^i \boldsymbol{\nu}_t' d\mathbf{B}_t + \frac{1}{\xi_t} \boldsymbol{\eta}_t^i d\mathbf{B}_t - \beta_t^{-1}\left(c_t^i - e_t^i\right) dt \\ &\quad + \widehat{W}_t^i \boldsymbol{\nu}_t' \boldsymbol{\nu}_t dt + \frac{1}{\xi_t} \boldsymbol{\nu}_t' \boldsymbol{\eta}_t^{i\prime} dt \end{aligned}$$

- Hence, using again $\mathbf{B}_t = \widehat{\mathbf{B}}_t - \int_0^t \boldsymbol{\nu}_u du$ we obtain

$$d\widehat{W}_t^i = -\beta_t^{-1}\left(c_t^i - e_t^i\right) dt + \frac{1}{\xi_t}\left(\boldsymbol{\eta}_t^i + \widehat{W}_t^i \boldsymbol{\nu}_t'\right) d\widehat{\mathbf{B}}_t \tag{15}$$

- Comparing (10) with (15), we then must have

$$\boldsymbol{\theta}_t^i \mathbf{I}_{\widehat{S}} \boldsymbol{\sigma}_t = \frac{1}{\xi_t} \left(\boldsymbol{\eta}_t^i + \widehat{W}_t^i \boldsymbol{\nu}_t' \right) = \frac{\boldsymbol{\eta}_t^i}{\xi_t} + \widehat{W}_t^i \boldsymbol{\nu}_t'$$

- Multiplying both sides by β_t we finally obtain

$$\boldsymbol{\theta}_t^i \mathbf{I}_S \boldsymbol{\sigma}_t = \frac{\boldsymbol{\eta}_t^i}{\pi_t} + W_t^i \boldsymbol{\nu}_t' \tag{16}$$

- where

$$W_t^i = \frac{1}{\pi_t} E_t \left(\int_t^T \pi_u \left(c_u^i - e_u^i \right) du \right) \tag{17}$$

- Using the definition

$$W_t^i = \theta_t^{0,i} \beta_t + \boldsymbol{\theta}_t^i \cdot \mathbf{S}_t$$

- we also find the allocation in bonds

$$\theta_t^{0,i} = \beta_t^{-1} \left(W_t^i - \boldsymbol{\theta}_t^i \cdot \mathbf{S}_t \right)$$

3 Equilibrium and the Representative Agent

- So far we merely repeated the exercise in TN1. Now, we impose market clearing conditions and obtain the equilibrium results
- In this section we are going to skip even more of the details.
- Let the aggregate endowment be denoted as

$$e = \sum_{i=1}^{m} e^i$$

- From the results in the previous section, we then have the following
- **Corollary 1**: In any equilibrium, we must have

$$e_t = \sum_{i=1}^{m} \mathcal{I}_u^i \left(\lambda_i e^{\phi t} \pi_t \right) \tag{18}$$

- where λ_i satisfy the system of equations

$$E \left[\int_0^T \pi_t \left(\mathcal{I}_u^i \left(\lambda_i e^{\phi t} \pi_t \right) - e_t^i \right) dt \right] = 0 \tag{19}$$

- The converse is also true: If there exists a vector $\boldsymbol{\lambda} = (\lambda_1, .., \lambda_m)$ such that (18) and (19) are satisfied, then the market is in equilibrium. In either case, the optimal consumption is given by

$$c_t^i = \mathcal{I}_u^i \left(\lambda_i e^{\phi t} \pi_t \right) \tag{20}$$

- The only part of the proof yet to determine is that if there exists $\lambda = (\lambda_1, .., \lambda_m)$ such that (18) and (19) are satisfied, then the resulting market is in equilibrium.
- This is true because
  1. If the vector exists, then we know that (20) maximizes utility.
  2. Hence, (18) implies that the commodity market is cleared
  $$\sum_{i=1}^{m} c^i = \sum_{i=1}^{m} e^i$$
  3. From the previous proof, recall that portfolio weights were determined by the martingale
  $$M_t^i = \int_0^t \boldsymbol{\eta}_u^i d\mathbf{B}_u = E_t\left[\int_0^T \pi_u \left(c_u^i - e_u^i\right) du\right]$$
     * From (19) we have $\sum_{i=1}^{n} M_t^i = 0$ which implies $\sum_{i=1}^{n} \eta^i = 0$.
     * Summing over (17) and then (16) we finally find
     $$\sum_{i=1}^{m} W_t^i = 0, \;\; \sum_{i=1}^{m} \theta_t^i = 0, \;\; \sum_{i=1}^{m} \theta_t^{0,i} = 0$$
- Equation (18) is key in the construction of a representative agent.
- Given the representative agent, we shall characterize the parameters of the process.
- Given a vector $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_m)$, let's define the function
$$\mathcal{I}_u\left(x, \boldsymbol{\lambda}\right) = \sum_{i=1}^{m} \mathcal{I}_u^i\left(\lambda_i x\right)$$

- This a continuous, decreasing function for $x \in (0, \infty)$.
- Define by $\mathcal{U}_c(.; \boldsymbol{\lambda})$ the inverse of $\mathcal{I}(.; \boldsymbol{\lambda})$, that is, that function such that for every x, we have

$$\mathcal{U}_c(\mathcal{I}_u(x, \boldsymbol{\lambda}), \boldsymbol{\lambda}) = x \tag{21}$$

- This function is strictly decreasing on $(0, \infty)$.
- Notice that the monotonicity of the function implies

$$\mathcal{I}_u(\mathcal{U}_c(c, \boldsymbol{\lambda}), \boldsymbol{\lambda}) = c$$

- Using these definitions, we can rewrite (18) as

$$e_t = \mathcal{I}_u\left(e^{\phi t}\pi_t, \boldsymbol{\lambda}\right)$$

- We can then invert this function to obtain a formula for the state-price density

$$\pi_t = e^{-\phi t}\mathcal{U}_c(e_t; \boldsymbol{\lambda}) \tag{22}$$

- The form of this state price density should look familiar.
- From corollary 1, we have
- **Corollary 2**: Under condition A, a financial market is in equilibrium if and only if its state-price density is given by π_t in (22) where $\boldsymbol{\lambda}$ satisfies the system of equations

$$E\left[\int_0^T e^{-\phi t}\mathcal{U}_c(e_t; \boldsymbol{\lambda})\left(\mathcal{I}_u^i(\lambda_i\mathcal{U}_c(e_t; \boldsymbol{\lambda})) - e_t^i\right)dt\right] = 0 \tag{23}$$

- In addition, consumption is given by

$$c_t^i = \mathcal{I}_u^i\left(\lambda_i \mathcal{U}_c\left(e_t; \boldsymbol{\lambda}\right)\right) \tag{24}$$

- Corollary 2 characterizes an equilibrium, if it exists.

3.1 The Representative Agent

- Define the function

$$\mathcal{U}\left(c; \boldsymbol{\lambda}\right) = \max_{\substack{c_1 \geq 0, .., c_m \geq 0: \\ \sum_{i=1}^m c_i = c}} \sum_{i=1}^m \frac{1}{\lambda_i} u^i\left(c_i\right) \tag{25}$$

- We then have

- **Proposition 1**: $\mathcal{U}\left(c; \boldsymbol{\lambda}\right)$ is a strictly increasing utility function satisfying condition A. In addition

$$\frac{d\mathcal{U}\left(c, \boldsymbol{\lambda}\right)}{dc} = \mathcal{U}_c\left(c, \boldsymbol{\lambda}\right)$$

- as defined in (21).

- The key point in the proof is to define

$$\hat{c}_t^i = \mathcal{I}_u^i\left(\lambda_i \mathcal{U}_c\left(c; \boldsymbol{\lambda}\right)\right) \tag{26}$$

- and show that effectively, these $\hat{c}_t^i$ satisfy (25). This is easy to see. Notice that (26) implies that

$$u_c^i\left(\hat{c}_t^i\right) = \lambda_i \mathcal{U}_c\left(c; \boldsymbol{\lambda}\right) \tag{27}$$

- Hence, consider now any other $(c_1, .., c_m)$ such that $\sum_{i=1}^{m} c_i = c$.
- From the strict concavity of the utility functions we have

$$\begin{aligned} \sum_{i=1}^{m} \frac{1}{\lambda_i} u^i\left(c^i\right) &\leq \sum_{i=1}^{m} \frac{1}{\lambda_i}\left(u^i\left(\hat{c}^i\right) + \left(c^i - \hat{c}^i\right) u_c^i\left(\hat{c}^i\right)\right) \\ &= \sum_{i=1}^{m} \frac{1}{\lambda_i} u^i\left(\hat{c}^i\right) + \mathcal{U}_c\left(c; \boldsymbol{\lambda}\right) \sum_{i=1}^{m}\left(c^i - \hat{c}^i\right) \\ &= \sum_{i=1}^{m} \frac{1}{\lambda_i} u^i\left(\hat{c}^i\right) \end{aligned}$$

- So, we have defined a representative agent who assign constant weights $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_m)$ to the various agents 1 to m.
- Given the result in (23) and (26), the weights λ_1,...,λ_m are chosen so that the optimal consumption of the representative agent equals the aggregate endowment.
- We finally notice the following homogeneity property of the aggregate (representative) utility function:
- For every constant α, we have

$$\begin{aligned} \mathcal{U}\left(c, \alpha\boldsymbol{\lambda}\right) &= \alpha^{-1}\mathcal{U}\left(c, \boldsymbol{\lambda}\right) \\ \mathcal{U}_c\left(c, \alpha\boldsymbol{\lambda}\right) &= \alpha^{-1}\mathcal{U}_c\left(c, \boldsymbol{\lambda}\right) \end{aligned}$$

- Since the marginal utility $\mathcal{U}_c\left(c, \alpha\boldsymbol{\lambda}\right)$ determines the state price density, this property is convenient to renormalize the state price density.

- **Theorem 1**: There exists a vector $\boldsymbol{\lambda}$ that satisfies the system (23). In addition, if for all i

$$\frac{-cu_{cc}(c)}{u_c(c)} \leq 1$$

this solution is "unique" in the sense that any other solution $\widehat{\boldsymbol{\lambda}}$ implies the existence of a constant α such that

$$\alpha\mathcal{U}_c(e_t, \boldsymbol{\lambda}) = \mathcal{U}_c\left(e_t, \widehat{\boldsymbol{\lambda}}\right)$$

3.2 Characterizing the Equilibrium

- We now have all the ingredients to solve back for the parameters of the price process and the interest rate process.

- First, notice that we can renormalize the state price density so that

$$\widehat{\pi}_0 = \mathcal{U}_c(e_0; \boldsymbol{\lambda}) = 1$$

- Define the process

$$\widehat{\xi}_t = \mathcal{U}_c(e_t; \boldsymbol{\lambda})$$

- By Ito's lemma we have

$$\widehat{\xi}_t = 1 + \int_0^t \mathcal{U}_{cc}(e_u; \boldsymbol{\lambda})\, de_u + \frac{1}{2}\int_0^t \mathcal{U}_{ccc}(e_u; \boldsymbol{\lambda})\,(de_u)^2$$

- If we assume that the aggregate endowment evolves according to the Ito process

$$de_t = e_t\mu_{e,t}dt + e_t\boldsymbol{\sigma}_{e,t}d\mathbf{B}_t$$

- we obtain

$$\widehat{\xi}_t = 1 + \int_0^t \left(\mathcal{U}_{cc}\left(e_u; \boldsymbol{\lambda}\right) e_u \mu_{e,u} + \frac{1}{2} \mathcal{U}_{ccc}\left(e_u; \boldsymbol{\lambda}\right) e_u^2 \boldsymbol{\sigma}_{e,u} \boldsymbol{\sigma}'_{e,u} \right) du + \int_0^t \mathcal{U}_{cc}\left(e_u; \boldsymbol{\lambda}\right) e_u \sigma_{e,u} d\mathbf{B}_u \qquad (28)$$

- Recall that from (22), we must have that the state price density is

$$\widehat{\pi}_t = e^{-\phi t} \widehat{\xi}_t$$

- On the other hand, we also have that given a system of prices $(\beta, \mathbf{S})$, each following a Ito's process as described at the beginning of the teaching notes, we must have that the state price density is

$$\pi_t = \beta_t^{-1} \xi_t$$

- where

$$\xi_t = \exp\left(-\frac{1}{2} \int_0^t \boldsymbol{\nu}_u \boldsymbol{\nu}'_u du - \int_0^t \boldsymbol{\nu}_u d\mathbf{B}_u \right)$$

- Notice that since $\xi_0 = 1$, ξ_t satisfies the integral equation

$$\xi_t = 1 - \int_0^t \xi_u \boldsymbol{\nu}'_u d\mathbf{B}_u$$

- Define

$$\widetilde{\xi}_t = \pi_t e^{\phi t} = \xi_t e^{\int_0^t (\phi - r_u) du}$$

- We then have

$$\tilde{\xi}_t = 1 + \int_0^t (\phi - r_u)\, \tilde{\xi}_u dt - \int_0^t \xi_u \boldsymbol{\nu}'_u d\mathbf{B}_u \tag{29}$$

- By definition, in equilibrium we must have

$$\widehat{\pi}_t = \pi_t$$

- or

$$\hat{\xi}_t = e^{\phi t}\widehat{\pi}_t = e^{\phi t}\pi_t = \tilde{\xi}_t$$

- Hence, equating the integral equations (28) and (29) we obtain the equalities

$$\begin{aligned} (\phi - r_t)\, \tilde{\xi}_t &= \mathcal{U}_{cc}(e_t; \boldsymbol{\lambda})\, e_t \mu_{e,t} + \frac{1}{2}\mathcal{U}_{ccc}(e_t; \boldsymbol{\lambda})\, e_t^2 \boldsymbol{\sigma}_{e,t}\boldsymbol{\sigma}'_{e,t} \\ \xi_t \boldsymbol{\nu}'_t &= -\mathcal{U}_{cc}(e_t; \boldsymbol{\lambda})\, e_t \sigma_{e,t} \end{aligned}$$

- Recall that by definition $\tilde{\xi}_t = \hat{\xi}_t = \mathcal{U}_c(e_t; \boldsymbol{\lambda})$, obtaining

$$\begin{aligned} r_t &= \phi - \frac{\mathcal{U}_{cc}(e_t; \boldsymbol{\lambda})\, e_t}{\mathcal{U}_c(e_t; \boldsymbol{\lambda})}\mu_{e,t} - \frac{1}{2}\frac{\mathcal{U}_{ccc}(e_t; \boldsymbol{\lambda})\, e_t^2}{\mathcal{U}_c(e_t; \boldsymbol{\lambda})}\boldsymbol{\sigma}_{e,t}\boldsymbol{\sigma}'_{e,t} \\ \boldsymbol{\nu}_t &= -\frac{\mathcal{U}_{cc}(e_t; \boldsymbol{\lambda})\, e_t}{\mathcal{U}_c(e_t; \boldsymbol{\lambda})}\sigma'_{e,t} \end{aligned}$$

- We can further define the *relative risk aversion of the representative agent* as

$$\gamma(e_t; \boldsymbol{\lambda}) = -\frac{\mathcal{U}_{cc}(e_t; \boldsymbol{\lambda})\, e_t}{\mathcal{U}_c(e_t; \boldsymbol{\lambda})}$$

- and the *relative prudence coefficient of the representative agent* as

$$q(e_t;\boldsymbol{\lambda}) = -\frac{\mathcal{U}_{ccc}(e_t;\boldsymbol{\lambda})\, e_t}{\mathcal{U}_{cc}(e_t;\boldsymbol{\lambda})}$$

- we can rewrite the equations as

$$\begin{aligned} r_t &= \phi + \gamma(e_t;\boldsymbol{\lambda})\,\mu_{e,t} - \frac{1}{2}\gamma(e_t;\boldsymbol{\lambda})\, q(e_t;\boldsymbol{\lambda})\,\boldsymbol{\sigma}_{e,t}\boldsymbol{\sigma}'_{e,t} \\ \boldsymbol{\nu}_t &= \gamma(e_t;\boldsymbol{\lambda})\,\sigma'_{e,t} \end{aligned}$$

- Therefore, we conclude that

  1. The risk-free rate increases linearly with the discount rate ϕ and the relative risk aversion coefficient $\gamma(e_t;\lambda)$, while it decreases with the relative prudence parameter $q(e_t;\boldsymbol{\lambda})$ and with the variance of the endowment process $\boldsymbol{\sigma}_{e,t}\boldsymbol{\sigma}'_{e,t}$;
  2. The market price of risk $\boldsymbol{\nu}_t$ increases linearly with the relative risk aversion coefficient and the variability of the endowment process $\boldsymbol{\sigma}'_{e,t}$ (recall that $\boldsymbol{\nu}_t$ is a vector).

- These results immediately imply the following equilibrium concept.

4 The Equity Premium and the Consumption CAPM

- From the definition of $\boldsymbol{\nu}_t$ we have

$$\boldsymbol{\sigma}_t \cdot \boldsymbol{\nu}_t = \boldsymbol{\mu}_t - r_t \mathbf{1}_d$$

- We then obtain

$$\boldsymbol{\mu}_t - r_t \mathbf{1}_d = \gamma\left(e_t; \boldsymbol{\lambda}\right) \boldsymbol{\sigma}_t \boldsymbol{\sigma}'_{e,t}$$

- That is

$$E\left[\frac{dS_t^i}{S_t^i}\right] - r_t dt = \gamma\left(e_t; \boldsymbol{\lambda}\right) \times Cov_t\left(\frac{dS_t^i}{S_t^i}, \frac{de_t}{e_t}\right) \tag{30}$$

- That is, the expected excess returns of asset i depends on the relative risk aversion and the covariance between asset i and the aggregate endowment process.
- There are two puzzles here:

1. Equation (30) must hold for the market as a whole.
    - The correlation between stock returns and consumption growth is anywhere between .12 and -.05 (see Campbell and Cochrane (1999), Table 7)
    - Even with the optimistic assumptions of correlation = .12, volatility of return = .17 and volatility of consumption growth =.02, we have expected excess return = .4% even with $\gamma = 10$.

- So, $\gamma = 100$ would be needed here to obtain an equity premium = 4%.
- Note that the risk free rate may actually end up being reasonable if γ is high enough: For instance, $\gamma = 100$ produces $r = 0$ if $\phi = .02$, $\mu_c = .02$ and $\sigma_e = .02$.
- The reason is that the precausionary saving motive $-1/2\gamma*(\gamma+1)\sigma_e^2$ kicks in.

2. The second puzzle is about the cross-section: The consumption CAPM does not seem to work.
   - Using the Fama French size/book-to-market portfolios as test portfolios, a Fama-MacBeth regression of returns on consumption growth yields an insignificant coefficient =.22 and cross-sectional $R^2 = 16\%$ (see Lettau and Ludvigson (2001, Table 3).
   - Recent papers however point at noise in the consumption data and they show that if one uses more lags and leads to compute consumption growth, the result works out.
   - Typical pitfal: Note that even if a Fama-MacBeth coefficient turns out to be positive and significant (and the R^2 is high) there is still an issue of economic significance: The γ needed to rationalize the result may be too high, as in the case of the aggregate market.

4.1 The CAPM

- From (30) it is possible to find a "beta" relationship.
- Define the $1 \times d$ process $\boldsymbol{\psi}_t$ using the relationship

$$\boldsymbol{\psi}_t \cdot \mathbf{I}_S \cdot \boldsymbol{\sigma}_t = e_t \cdot \boldsymbol{\sigma}_{e,t}$$

- We can consider ψ_t as a self-financing strategy $\left(\theta^0, \boldsymbol{\theta}\right)$ by choosing $\boldsymbol{\theta}_t^i = \psi_t^i$ and θ_t^0 to meet the self-financing constraint.
- Let S_t^ψ be the value of the portfolio $S_t^\psi = \boldsymbol{\psi}_t \cdot S_t$. We then have

$$\frac{dS_t^\psi}{S_t^\psi} = \mu_{\psi,t}dt + \boldsymbol{\sigma}_{\psi,t}d\mathbf{B}_t$$

- where

$$\boldsymbol{\sigma}_{\psi,t} = \frac{1}{S_t^\psi}\boldsymbol{\psi}_t \cdot \mathbf{I}_S \cdot \boldsymbol{\sigma}_t = \frac{e_t}{S_t^\psi}\boldsymbol{\sigma}_{e,t}$$

- and $\mu_{\psi,t}$ satisfies the condition (30)

$$\mu_{\psi,t} - r_t = \gamma\left(e_t; \boldsymbol{\lambda}\right) \times \boldsymbol{\sigma}_{\psi,t}\sigma'_{e,t} \tag{31}$$

- Hence, we finally have that for all $i = 1, .., n$

$$\begin{aligned} \mu_t^i - r_t &= \gamma\left(e_t; \boldsymbol{\lambda}\right) \boldsymbol{\sigma}_t^i \boldsymbol{\sigma}'_{e,t} \\ &= \gamma\left(e_t; \boldsymbol{\lambda}\right) \times \frac{S_t^\psi}{e_t} \times \sigma_t^i \sigma'_{\psi,t} \end{aligned}$$

- Since this holds for the portfolio ψ as well, that is

$$\mu_t^\psi - r_t = \gamma\left(e_t; \boldsymbol{\lambda}\right) \times \frac{S_t^\psi}{e_t} \times \sigma_{\psi,t}\sigma'_{\psi,t}$$

- by substituting for the common term $\gamma\left(e_t;\boldsymbol{\lambda}\right)\times\frac{S_t^{\psi}}{e_t}$ we obtain the CAPM "beta" relationship

$$E_t\left(\frac{dS_t^i}{S_t^i}\right)-r_t dt=\beta_t^i\times\left(E_t\left(\frac{dS_t^{\psi}}{S_t^{\psi}}\right)-r_t\mathbf{1}_d\right)$$

- where

$$\beta_t^i=\frac{\boldsymbol{\sigma}_t^i\cdot\boldsymbol{\sigma}'_{\psi,t}}{\boldsymbol{\sigma}_{\psi,t}\cdot\boldsymbol{\sigma}'_{\psi,t}}=\frac{cov_t\left(dS_t^i/S_t^i,dS_t^{\psi}/S_t^{\psi}\right)}{var_t\left(dS_t^{\psi}/S_t^{\psi}\right)}$$

- **Note:**
  - The (conditional) CAPM works with respect to that asset that is perfectly correlated with the endowment process (and thus the stochastic discount factor).
  - This need not be the market portfolio. The existence of labor income, for instance, generate a wedge between the stochastic discount factor and the market portfolio: Thus, the CAPM is not supposed to be working (theoretically) with respect to the market portfolio.
  - But even in the case where there is no labor income it is not obvious that the market portfolio is perfectly correlated with the endowment process.
  - We will do more on this later on in the course.

4.2 An Asset Pricing Relationship

- Suppose that agents i sells a financial asset whose payment is its entire endowment (or a fraction of it), that is e_t^i.
- What is the fair price of this stream of payments?
- The above results imply that the price at time t of this claim is

$$S_t^{e^i} = \frac{1}{\pi_t} E_t \left[\int_t^T \pi_u e_u^i du \right] \qquad (32)$$

$$= \frac{1}{\mathcal{U}_c(e_t)} E_t \left[\int_t^T e^{-\phi(u-t)} \mathcal{U}_c(e_u) e_u^i du \right] \qquad (33)$$

- In fact, if this was not true one could find a trading strategy $(\theta_0, \boldsymbol{\theta})$ that finances e_u^i and whose value is (32). This in turn generates an arbitrage opportunity.
- Indeed, the value of total endowment process is simply

$$S_t = \frac{1}{\mathcal{U}_c(e_t)} E_t \left[\int_t^T e^{-\phi(u-t)} \mathcal{U}_c(e_u) e_u du \right] \qquad (34)$$

- We will use this pricing equation often.

4.3 A simple example: Log Utility

- Suppose all the agents have logarithmic utility

$$u^i(c_t) = \log(c_t)$$

- so that the inverse of the marginal utility is given by

$$\mathcal{I}_u^i(x) = x^{-1}$$

- In this case, the marginal utility of the representative investor is

$$\mathcal{U}_c(c) = \frac{1}{c}\sum_{i=1}^{m} \lambda_i^{-1}$$

- We can renormalize the weights so that $\sum_{i=1}^{m} \lambda_i^{-1} = e_0$, which we can assume positive.
- We then have

$$\mathcal{U}_c(e_0) = 1$$

- The vector $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_m)$ has to satisfy the system of equations (23),

$$E\left[\int_0^T e^{-\phi t}\mathcal{U}_c(e_t; \boldsymbol{\lambda})\left(\mathcal{I}_u^i(\lambda_i \mathcal{U}_c(e_t; \boldsymbol{\lambda})) - e_t^i\right)dt\right] = 0 \quad (35)$$

- which becomes

$$E\left[\int_0^T e^{-\phi t}\frac{e_0}{e_t}\left(\left(\lambda_i \frac{e_0}{e_t}\right)^{-1} - e_t^i\right)dt\right] = 0 \quad (36)$$

- and in turn

$$\lambda_i^{-1} = e_0 \frac{E\left[\int_0^T e^{-\phi t} \frac{e_t^i}{e_t} dt\right]}{\int_0^T e^{-\phi t} dt} \tag{37}$$

- Hence

$$\pi_t = e^{-\phi t} \mathcal{U}_c\left(e_t; \boldsymbol{\lambda}\right) = e^{-\phi t} \frac{e_0}{e_t}$$

- We then have that

$$c_t^i = \mathcal{I}_u\left(\lambda_i e^{\phi t} \pi_t\right) = \mathcal{I}_u\left(\lambda_i \frac{e_0}{e_t}\right) = \lambda_i^{-1} \frac{e_t}{e_0} \tag{38}$$

- Finally, we have

$$\begin{aligned} \gamma\left(e_t; \boldsymbol{\lambda}\right) &= -\frac{e_t \mathcal{U}_{cc}\left(e_t, \lambda\right)}{\mathcal{U}_c\left(e_t, \lambda\right)} = -\frac{e_t\left(-e_0/e_t^2\right)}{e_0/e_t} = 1 \\ q\left(e_t; \boldsymbol{\lambda}\right) &= -\frac{e_t \mathcal{U}_{ccc}\left(e_t, \lambda\right)}{\mathcal{U}_{cc}\left(e_t, \lambda\right)} = -\frac{e_t 2 e_0/e_t^3}{\left(-e_0/e_t^2\right)} = 2 \end{aligned}$$

- Hence, the condition for the market equilibrium are

$$\begin{aligned} r_t &= \phi + \mu_{e,t} - \boldsymbol{\sigma}_{e,t} \boldsymbol{\sigma}'_{e,t} \\ \boldsymbol{\nu}_t &= \sigma'_{e,t} \end{aligned}$$

4.4 An Alternative Formula for the C-CAPM

- Before commenting further, notice also that an alternative expression for the (30) can be obtained as follows.

- We know that at an optimum, the following condition holds (see (27))

$$u_c^j\left(\widehat{c}_t^j\right) = \lambda_j \mathcal{U}_c\left(e; \boldsymbol{\lambda}\right) \tag{39}$$

- Hence, rather than using the representative agent utility function, we may think of using the marginal utility of agent j defined on the optimal consumption path $\widehat{c}^j$.
- Equation (39) ensures that the two approaches are identical.
- Define the process

$$\widehat{\xi}_t^j = \frac{1}{\lambda_j} u_c^j\left(\widehat{c}_t^j\right)$$

- Define the state-price density of agent j as

$$\widehat{\pi}_t^j = \beta_t^{-1} \widehat{\xi}_t^j$$

- By going through the same typo of calculation, it is clear that everywhere we can substitute the "representative agent" utility and endowment, with agent i utility and *consumption.*
- As a consequence, on then obtains

$$\begin{aligned} E\left[\frac{dS_t^i}{S_t^i}\right] - r_t dt &= \gamma^j\left(\widehat{c}_t^j\right) \times Cov_t\left(\frac{dS_t^i}{S_t^i}, \frac{d\widehat{c}_t^j}{\widehat{c}_t^j}\right) \\ &= a^j\left(\widehat{c}_t^j\right) \times Cov_t\left(\frac{dS_t^i}{S_t^i}, d\widehat{c}_t^j\right) \end{aligned}$$

- where

$$a^j\left(\widehat{c}_t^j\right) = -\frac{u_{cc}^j\left(\widehat{c}_t^j\right)}{u_c^j\left(\widehat{c}_t^j\right)}$$

- is the coefficient of absolute risk aversion of agent j.
- Divide now both sides by $a^j\left(\widehat{c}_t^j\right)$ and sum across $j = 1, ..., m$. Since from the market clearing condition $\sum_{j=1}^n \widehat{c}_t^j = e_t$ we must have

$$\sum_{j=1}^m Cov_t\left(\frac{dS_t^i}{S_t^i}, d\widehat{c}_t^j\right) = Cov_t\left(\frac{dS_t^i}{S_t^i}, d\widehat{c}_t^j\right)$$

- we obtain

$$E\left[\frac{dS_t^i}{S_t^i}\right] - r_t dt = \Gamma\left(\mathbf{c}_t\right) \times Cov_t\left(\frac{dS_t^i}{S_t^i}, de_t\right)$$

- The coefficient

$$\Gamma\left(\mathbf{c}_t\right) = \frac{1}{\sum_{j=1}^m a^j\left(\widehat{c}_t^j\right)^{-1}}$$

- is the coefficient of absolute risk aversion of the market itself.