

Topics in Dynamic Asset Pricing

Teaching Notes 1 (Addendum 2).

An Example of Portfolio Selection
with time-varying opportunity set
in Complete Markets

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The Portfolio Problem

- Consider the following portfolio problem with time-varying expected returns.
- Let (β, \mathbf{S}) be processes of the form

$$d\beta_t = r_t\beta_t dt \text{ with } \beta_0 > 0 \quad (1)$$

$$dS_t^i = \mu_t^i S_t^i dt + S_t^i \sigma_t^i d\mathbf{B}_t \text{ with } S_0^i > 0 \quad (2)$$

- where r_t , $\boldsymbol{\mu}_t = (\mu_t^1, \dots, \mu_t^d)$ and $\boldsymbol{\sigma}^i$ are bounded, adapted processes.
- For concreteness, assume that $\mu_t = (\mu_t^1, \dots, \mu_t^d)$ follow a continuous time, VAR process

$$d\boldsymbol{\mu}_t = (\mathbf{A}_0 + \mathbf{A}_1\boldsymbol{\mu}_t) dt + \boldsymbol{\Sigma}d\mathbf{B}_t$$

- Notice that the set of Brownian motions moving $\boldsymbol{\mu}_t$ is the same as the ones moving S_t .
- Instead, assume $r_t = r$ is constant and that $\boldsymbol{\sigma}_t^i = \boldsymbol{\sigma}$ is also constant, for every i .
- Assume again that $\{\boldsymbol{\sigma}\}$ is invertible.

The dynamics of the market price of risk

- Define by $\lambda_t = \mu_t - r\mathbf{1}_d$ the excess return process.
- Let the $d \times 1$ market price of risk process

$$\nu_t = \sigma^{-1}\lambda_t$$

- Clearly, also ν_t follows a VAR process

$$\begin{aligned} d\nu_t &= \sigma^{-1}d\mu_t \\ &= (\sigma^{-1}\mathbf{A}_0 + \sigma^{-1}\mathbf{A}_1\mu_t) dt + \sigma^{-1}\Sigma d\mathbf{B}_t \\ &= (\tilde{\mathbf{A}}_0 + \tilde{\mathbf{A}}_1\nu_t) dt + \tilde{\Sigma}d\mathbf{B}_t \end{aligned}$$

- with $\tilde{\mathbf{A}}_0 = \sigma^{-1}(\mathbf{A}_0 + \mathbf{A}_1\mathbf{1}_d r)$, $\tilde{\mathbf{A}}_1 = \sigma^{-1}\mathbf{A}_1\sigma$ and $\tilde{\Sigma} = \sigma^{-1}\Sigma$.

- **Result:** Given an initial condition $\boldsymbol{\nu}_0 = \widehat{\boldsymbol{\nu}}$, then for $\tau > 0$

$$\boldsymbol{\nu}_\tau \sim N(\boldsymbol{\alpha}(\boldsymbol{\nu}_0, \tau), \mathbf{S}(\tau))$$

- where

$$\begin{aligned}\boldsymbol{\alpha}(\boldsymbol{\nu}_0, \tau) &= \boldsymbol{\Psi}(\tau) \boldsymbol{\nu}_0 + \boldsymbol{\zeta}(\tau) \\ \mathbf{S}(\tau) &= \int_0^\tau \boldsymbol{\Psi}(\tau - s) \widetilde{\boldsymbol{\Sigma}} \widetilde{\boldsymbol{\Sigma}}' \boldsymbol{\Psi}(\tau - s)' ds \\ \boldsymbol{\zeta}(\tau) &= \int_0^\tau \boldsymbol{\Psi}(\tau - s) \widetilde{\mathbf{A}}_0 ds\end{aligned}$$

- and $\boldsymbol{\Psi}(\tau)$ solves the system of differential equation

$$\frac{d\boldsymbol{\Psi}(t)}{dt} = \widetilde{\mathbf{A}}_1 \boldsymbol{\Psi}(t)$$

with initial condition $\boldsymbol{\Psi}(0) = \mathbf{I}$.

- If $\widetilde{\mathbf{B}}$ has distinct and real eigenvalues, then the solution is

$$\boldsymbol{\Psi}(\tau) = \mathbf{U} \exp(\boldsymbol{\Lambda} \cdot \tau) \mathbf{U}^{-1}$$

where, $\boldsymbol{\Lambda}$ is the diagonal matrix with $\widetilde{\boldsymbol{\Lambda}}_1$ eigenvalues on the principal diagonal, \mathbf{U} is the matrix of the associated eigenvectors, and $\exp(\boldsymbol{\Lambda} \cdot T)$ is the diagonal matrix with $e^{\lambda_i T}$ in its ii -th position.

- Clearly, we need $\lambda_i \leq 0$ to ensure that the solution does not explode.

- In this case, we have that the Novikov's condition is satisfied:

$$E \left[\exp \left(\frac{1}{2} \int_0^T \boldsymbol{\nu}'_t \boldsymbol{\nu}_t dt \right) \right] < \infty$$

- Thus,

$$\xi_t = \exp \left(-\frac{1}{2} \int_0^t \boldsymbol{\nu}'_u \boldsymbol{\nu}_u du - \int_0^t \boldsymbol{\nu}'_u d\mathbf{B}_u \right)$$

defines a P-martingale.

The Optimal Consumption Plan

- Define the *state-price deflator*

$$\pi_t = e^{-rt} \xi_t = \exp \left(- \left(\int_0^t r + \frac{1}{2} \boldsymbol{\nu}'_u \boldsymbol{\nu}_u du \right) - \int_0^t \boldsymbol{\nu}'_u d\mathbf{B}_u \right) \quad (3)$$

- We then have that the optimal consumption is given by

$$C_t^* = \mathcal{I}_u (\lambda \pi_t, t) \quad (4)$$

- where \mathcal{I}_u is the inverse of the utility functions.
- In addition, by defining again

$$\widehat{w}(\lambda) = E \left(\int_0^T \pi_t \mathcal{I}_u (\lambda \pi_t, t) dt \right) \quad (5)$$

- the solution to λ^* is given by the equality $\widehat{w}(\lambda) = w$.
- Assume for instance a power utility

$$u(C, t) = e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma}$$

- This implies

$$u_c = e^{-\rho t} C_t^{-\gamma} \quad \implies \quad \mathcal{I}_u(x, t) = e^{-\frac{\rho}{\gamma} t} x^{-\frac{1}{\gamma}}$$

- Hence

$$\begin{aligned} C_t^* &= e^{-\frac{\rho}{\gamma} t} (\lambda \pi_t)^{-\frac{1}{\gamma}} \\ &= e^{-\frac{\rho}{\gamma} t} \left(\lambda \exp \left(- \int_0^t r + \frac{1}{2} \boldsymbol{\nu}'_u \boldsymbol{\nu}_u du - \int_0^t \boldsymbol{\nu}'_u d\mathbf{B}_u \right) \right)^{-\frac{1}{\gamma}} \\ &= \lambda^{-\frac{1}{\gamma}} \exp \left(\int_0^t \frac{1}{\gamma} (r - \rho) + \frac{1}{2\gamma} \boldsymbol{\nu}'_u \boldsymbol{\nu}_u du + \frac{1}{\gamma} \int_0^t \boldsymbol{\nu}'_u d\mathbf{B}_u \right) \end{aligned}$$

- where λ is a constant determined by the budget constraint.
- The process for log consumption $c_t = \log(C_t)$ is then given by

$$c_t = -\frac{1}{\gamma} \log(\lambda) + \left(\int_0^t \frac{1}{\gamma} (r - \rho) + \frac{1}{2\gamma} \boldsymbol{\nu}'_u \boldsymbol{\nu}_u du + \frac{1}{\gamma} \int_0^t \boldsymbol{\nu}'_u d\mathbf{B}_u \right)$$

- implying

$$dc_t = \left(\frac{1}{\gamma} (r - \rho) + \frac{1}{2\gamma} \boldsymbol{\nu}'_t \boldsymbol{\nu}_t \right) dt + \frac{1}{\gamma} \boldsymbol{\nu}'_t d\mathbf{B}_t$$

The Optimal Portfolio Weights

- Consider now all the processes under Q : Define the new Brownian motion

$$d\widehat{\mathbf{B}}_t = d\mathbf{B}_t + \boldsymbol{\nu}_t dt$$

- so that, under Q , we have

$$dc_t = \left(\frac{1}{\gamma} (r - \rho) - \frac{1}{2\gamma} \boldsymbol{\nu}'_t \boldsymbol{\nu}_t \right) dt + \frac{1}{\gamma} \boldsymbol{\nu}'_t d\widehat{\mathbf{B}}_t \quad (6)$$

$$d\boldsymbol{\nu}_t = \left(\widetilde{\mathbf{A}}_0 + \widehat{\mathbf{A}}_1 \boldsymbol{\nu}_t \right) dt + \widetilde{\boldsymbol{\Sigma}} d\widehat{\mathbf{B}}_t$$

- with $\widehat{\mathbf{A}}_1 = \widetilde{\mathbf{A}}_1 - \widetilde{\boldsymbol{\Sigma}}$.
- We now find the portfolio weights that support C_t^* .
- First, define the Q -martingale

$$M_t = E^Q \left[\int_0^T \beta_u^{-1} C_u du \right] \quad (7)$$

- We know that $M_0 = w =$ wealth at time 0.
- From the martingale representation theorem, we know that there exists $\widehat{\boldsymbol{\eta}}_t$ such that

$$dM_t = \widehat{\boldsymbol{\eta}}_t d\widehat{\mathbf{B}}_t$$

- The (discounted) wealth is given by

$$\widehat{W}_t = \beta_t^{-1} W_t = E_t^Q \left[\int_t^T \beta_u^{-1} C_u du \right] = M_t - J_t \quad (8)$$

- where

$$J_t = \int_0^t \beta_u^{-1} C_u du$$

- Thus, the process for the discounted wealth is

$$d\widehat{W}_t = -\beta_t^{-1} C_t dt + \widehat{\eta}_t d\widehat{\mathbf{B}}_t \quad (9)$$

- We also have that the wealth is always equal to the total amount invested in stocks and bonds, which must satisfy the budget constraint

$$\widehat{W}_t = \theta_t^0 + \boldsymbol{\theta}_t \widehat{\mathbf{S}}_t = \int_0^t \boldsymbol{\theta}_u d\widehat{\mathbf{S}}_t - \int_0^t \beta_u^{-1} C_u du$$

- where $\widehat{\mathbf{S}}_t = \beta_t^{-1} \mathbf{S}_t$ is a martingale under Q

$$d\widehat{\mathbf{S}}_t = \mathbf{I}_{\widehat{\mathbf{S}}} \boldsymbol{\sigma} d\widehat{\mathbf{B}}_t$$

- Thus, the process for the discounted wealth under Q is

$$d\widehat{W}_t = -\beta_t^{-1} C_t + \boldsymbol{\theta}_t \mathbf{I}_{\widehat{\mathbf{S}}} \boldsymbol{\sigma} d\widehat{\mathbf{B}}_t$$

- which, comparing with (9), yields immediately

$$\boldsymbol{\theta}_t \mathbf{I}_{\widehat{\mathbf{S}}} \boldsymbol{\sigma} = \widehat{\eta}_t$$

How do we get $\widehat{\eta}_t$ practically?

- Notice that we can rewrite (8) as

$$\begin{aligned}\widehat{W}_t &= E_t^Q \left[\int_t^T \beta_u^{-1} C_u du \right] \\ &= \beta_t^{-1} C_t E_t^Q \left[\int_t^T \frac{\beta_u^{-1} C_u}{\beta_t^{-1} C_t} du \right] \\ &= \beta_t^{-1} C_t E_t^Q \left[\int_t^T \frac{\beta_u^{-1}}{\beta_t^{-1}} e^{c_u - c_t} du \right]\end{aligned}$$

- From the process for optimal consumption (6), the conditional expectation $E_t^Q \left[\int_t^T \frac{\beta_u^{-1}}{\beta_t^{-1}} e^{c_u - c_t} du \right]$ depends only on $\boldsymbol{\nu}_t$.
- In other words, we can define the function

$$F(\boldsymbol{\nu}_t, t; T) = E_t^Q \left[\int_t^T \frac{\beta_\tau^{-1}}{\beta_t^{-1}} e^{c_\tau - c_t} du \right] \quad (10)$$

- and therefore the process

$$\widehat{W}_t = \beta_t^{-1} C_t F(\boldsymbol{\nu}_t, t; T)$$

- Thus, using Ito's lemma, the diffusion part of the discounted wealth process $d\widehat{W}_t$ must be given by

$$\widehat{\sigma}'_W = \widehat{W}_t \left(\frac{1}{\gamma} \boldsymbol{\nu}'_t + \frac{1}{F} \sum_{i=1}^n \frac{\partial F}{\partial \nu^i} \widetilde{\sigma}^i \right)$$

- with $\widetilde{\sigma}^i = \left[\widetilde{\Sigma} \right]_i$, the i -th row of $\widetilde{\Sigma}$.
- In other words, from (9) we must have $\widehat{\eta}_t = \widehat{\sigma}'_W$.
- This yields

$$\boldsymbol{\theta}_t \mathbf{I}_{\widehat{S}} \boldsymbol{\sigma} = \widehat{W}_t \left(\frac{1}{\gamma} \boldsymbol{\nu}'_t + \frac{1}{F} \sum_{i=1}^n \frac{\partial F}{\partial \nu^i} \widetilde{\sigma}^i \right)$$

- or

$$\boldsymbol{\theta}_t \mathbf{I}_{\widehat{S}} = \widehat{W}_t \left(\frac{1}{\gamma} \boldsymbol{\nu}'_t + \frac{1}{F} \sum_{i=1}^n \frac{\partial F}{\partial \nu^i} \widetilde{\sigma}^i \right) \boldsymbol{\sigma}^{-1}$$

Myopic and Hedging Demand

- In terms of fraction of wealth, and recalling $\boldsymbol{\nu}_t = \boldsymbol{\sigma}^{-1} (\boldsymbol{\mu}_t - r\mathbf{1}_n)$

$$\boldsymbol{\vartheta}'_t = \frac{\boldsymbol{\theta}_t \mathbf{I}_{\widehat{S}}}{\widehat{W}_t} = \frac{\boldsymbol{\theta}_t \mathbf{I}_S}{W_t} = \frac{(\boldsymbol{\mu}_t - r\mathbf{1}_n)'}{\gamma} (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1} + \sum_{i=1}^n \frac{1}{F} \frac{\partial F}{\partial \nu^i} \tilde{\boldsymbol{\sigma}}^i \boldsymbol{\sigma}^{-1} \quad (11)$$

- The first term on the RHS is the usual “myopic term”: Higher excess return increase the portfolio holding, while higher risk and higher risk aversion decreases it.
- The second term is the hedging demand component. Notice first we can write

$$\tilde{\boldsymbol{\sigma}}^i \boldsymbol{\sigma}^{-1} = \left(\tilde{\boldsymbol{\sigma}}^i \boldsymbol{\sigma} \right) (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}$$

- In addition, it turns out (see later) that

$$\frac{1}{F} \frac{\partial F}{\partial \nu^i} = - \frac{J_{W\nu^i}}{W J_{WW}}$$

- where $J(W, t; T) = E_t \left[e^{-\rho(\tau-t)} \frac{C_\tau^{1-\gamma}}{1-\gamma} \right]$ is the indirect utility function.
- Thus, the hedging demand is given by

$$\text{Hedging Demand} = - \sum_{i=1}^n \frac{J_{W\nu^i}}{W J_{WW}} \left(\tilde{\boldsymbol{\sigma}}^i \boldsymbol{\sigma} \right) (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}$$

- I give an intuition of this term below in the context of a specific example.

How do we compute $F(\boldsymbol{\nu}_t, t; T)$ in (10) ?

- From contingent claim valuation: Consider a security that pays out C_t over time as dividend.
- Under Q, its value is

$$V(C_t, \boldsymbol{\nu}_t, t; T) = E_t^Q \left[\int_t^T e^{-r(\tau-t)} C_\tau d\tau \right] = C_t F(\boldsymbol{\nu}_t, t; T)$$

- The total expected return (under Q) on this security must equal the risk free rate, so that

$$E_t^Q [dV] + C dt = V r dt \quad (12)$$

- From Ito's Lemma

$$\begin{aligned} E_t^Q [dV] &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial C} C_t \mu_C(\boldsymbol{\nu}_t) + \sum_{i=1}^d \frac{\partial V}{\partial \nu^i} \mu_i(\boldsymbol{\nu}_t) \\ &\quad + \sum_{i=1}^d \frac{\partial^2 V}{\partial C \partial \nu^i} C_t \frac{\boldsymbol{\nu}_t}{\gamma} \tilde{\boldsymbol{\sigma}}_i + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2 V}{\partial \nu^i \partial \nu^j} \tilde{\boldsymbol{\sigma}}_i \tilde{\boldsymbol{\sigma}}_j' \end{aligned}$$

- where

$$\mu_C(\boldsymbol{\nu}_t) = \frac{1}{\gamma} (r - \rho) + \frac{\boldsymbol{\nu}_t' \boldsymbol{\nu}_t}{2\gamma} \left(\frac{1}{\gamma} - 1 \right) \quad (13)$$

$$\mu_i(\boldsymbol{\nu}_t) = \tilde{\mathbf{A}}_{0,i} + \hat{\mathbf{A}}_{1,i} \boldsymbol{\nu}_t \quad (14)$$

- Finally, since

$$\frac{\partial V}{\partial t} = C \frac{\partial F}{\partial t}; \quad \frac{\partial V}{\partial C} = F; \quad \frac{\partial V}{\partial \nu^i} = C \frac{\partial F}{\partial \nu^i}; \quad \frac{\partial^2 V}{\partial C \partial \nu^i} = \frac{\partial F}{\partial \nu^i}; \quad \frac{\partial^2 V}{\partial \nu^i \partial \nu^j} = \frac{\partial^2 F}{\partial \nu^i \partial \nu^j} C$$

- substituting everything into (12), we find

$$F(r - \mu_C(\boldsymbol{\nu}_t)) = 1 + \frac{\partial F}{\partial t} + \sum_{i=1}^d \frac{\partial F}{\partial \nu^i} \left(\tilde{\mathbf{A}}_{0,i} + \left(\hat{\mathbf{A}}_{1,i} + \frac{\tilde{\boldsymbol{\sigma}}_i}{\gamma} \right) \boldsymbol{\nu}_t \right) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2 F}{\partial \nu^i \partial \nu^j} \tilde{\boldsymbol{\sigma}}_i \tilde{\boldsymbol{\sigma}}_j' \quad (15)$$

- with final condition

$$F(\boldsymbol{\nu}_T, T; T) = 0.$$

- If a solution is not available, the PDE can often be computed numerically.
- From F , one can then obtain $\boldsymbol{\eta}_t$ and, in addition, also consumption. In fact, recall that

$$\widehat{W}_t = \beta_t^{-1} C_t F(\boldsymbol{\nu}_t, t; T)$$

- which gives

$$C_t = W_t F(\boldsymbol{\nu}_t, t; T)^{-1}$$

The Hedging Demand Again

- We can finally show that

$$\frac{1}{F} \frac{\partial F}{\partial \nu^i} = - \frac{J_{W\nu^i}}{W J_{WW}} \quad (16)$$

- From the Bellman equation (see TN 1), we have that the first order conditions with respect to consumption C is

$$J_W = u_c(C_t^*)$$

- where $C_t^* = C(W_t, \nu_t) = W_t F(\nu_t, t; T)^{-1}$ is the optimal policy function.
- Differentiating both sides with respect to W and, say, ν^i yields the equalities

$$J_{WW} = u_{cc} \frac{\partial C^*}{\partial W} = F(\nu_t, t; T)^{-1}$$

$$J_{W\nu^i} = u_{cc} \frac{\partial C^*}{\partial \nu^i} = -W F(\nu_t, t; T)^{-2} \frac{\partial F}{\partial \nu^i}$$

- It is immediate to verify that the RHS and LHS of (16) coincide.

The case $\gamma = 1$

- Notice that if $\gamma = 1$, then $F(\nu_t, t; T) = F(t; T)$ is the solution to the PDE.
- In fact, in this case, we obtain

$$F(t; T)\rho = 1 + F'(t; T)$$

which yields

$$F(t; T) = He^{\rho t} + \frac{1}{\rho}$$

- The final condition

$$F(T; T) = He^{\rho T} + \frac{1}{\rho} = 0$$

- yields

$$H = \frac{e^{-\rho T}}{\rho}$$

- Thus

$$F(t; T) = \frac{1}{\rho} \left(1 - e^{-\rho(T-t)} \right)$$

- This implies that the consumption to wealth ratio is deterministic and given by

$$C_t = \frac{W_t \rho}{(1 - e^{-\rho(T-t)})}$$

- In addition, the optimal portfolio choice is

$$\boldsymbol{\vartheta}'_t = \frac{(\boldsymbol{\mu}_t - r \mathbf{1}_n)'}{\gamma} (\boldsymbol{\sigma} \boldsymbol{\sigma}')^{-1}$$

The Solution in the Univariate Case for $\gamma > 1$

- Consider the univariate case ($d = 1$).
- In this case, the portfolio holding of the market is given by

$$\vartheta_t = \frac{\mu_t - r}{\gamma\sigma^2} + \left(\frac{1}{F} \frac{\partial F}{\partial \nu} \right) \frac{\tilde{\sigma}}{\sigma}$$

- where F has to satisfy the PDE (15) becomes

$$F(r - \mu_C(\nu_t)) = 1 + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \nu} \left(\tilde{A}_0 + \left(\hat{A}_1 + \frac{\tilde{\sigma}}{\gamma} \right) \nu_t \right) + \frac{1}{2} \frac{\partial^2 F}{\partial \nu^2} \tilde{\sigma}^2 \quad (17)$$

- Unfortunately, it does not have a closed form solution.
- Wachter (2002) however finds a nice way out: Rather than considering a claim to the whole sequence of consumption plan $\{C_\tau\}_t^T$, that is, the security

$$V(C, \nu, t; T) = E^Q \left[\int_0^T e^{-r(\tau-t)} C_\tau d\tau \right]$$

she first considers a set of claims, each to the exact consumption “coupon” C_τ paid at that particular τ , for $\tau \in [0, T]$.

- By no arbitrage, the value of the security paying the process $\{C_\tau\}$ will be the sum (i.e. the integral) of all these individual claims.
- Let the value of each claim to the “coupon” C_τ be given by

$$v(C, \nu, t; \tau) = E^Q \left[e^{-r(\tau-t)} C_\tau \right].$$

- The homogeneity discussed earlier entails that $v(C, \nu, t; \tau) = C f(\nu, t; \tau)$ for some $f(\cdot)$.
- Under Q also this claim must earn the risk free rate

$$E^Q [dv] = rv dt$$

- Ito's Lemma then gives

$$rv = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial C} C_t \mu_C(\nu_t) + \frac{\partial v}{\partial \nu} \left(\hat{A}_0 + \hat{A}_1 \nu_t \right) + \frac{1}{2} \frac{\partial^2 v}{\partial \nu^2} \hat{\sigma}^2 + \frac{\partial^2 v}{\partial C \partial \nu} C \frac{\nu_t}{\gamma} \hat{\sigma}$$

- with final condition $v(C_\tau, \nu_\tau, \tau; \tau) = C_\tau$
- Recall also that under Q

$$\mu_C(\nu_t) = \frac{1}{\gamma} (r - \rho) + \frac{\nu_t^2}{2\gamma} \left(\frac{1}{\gamma} - 1 \right)$$

- As before, we can substitute the following quantities

$$\frac{\partial v}{\partial t} = C \frac{\partial f}{\partial t}; \quad \frac{\partial v}{\partial C} = f; \quad \frac{\partial v}{\partial \nu} = C \frac{\partial f}{\partial \nu}; \quad \frac{\partial^2 v}{\partial \nu^2} = C \frac{\partial^2 f}{\partial \nu^2}; \quad \frac{\partial^2 v}{\partial C \partial \nu} = \frac{\partial f}{\partial \nu}$$

- Substituting also $v = Cf$, $\mu_C(\nu)$ and deleting C on both sides yields

$$rf = \frac{\partial f}{\partial t} + f \left(\frac{1}{\gamma}(r - \rho) + \frac{\nu_t^2}{2\gamma} \left(\frac{1}{\gamma} - 1 \right) \right) + \frac{\partial f}{\partial \nu} \left(\hat{A}_0 + \hat{A}_1 \nu_t \right) + \frac{1}{2} \frac{\partial^2 f}{\partial \nu^2} \hat{\sigma}^2 + \frac{\partial f}{\partial \nu} \frac{\nu_t}{\gamma} \hat{\sigma}$$

- with final condition $f(\nu_\tau, \tau; \tau) = 1$
- Fortunately, a “solution” to this PDE instead exists.
- **How can we find it constructively?**
- Use the **method of undetermined coefficients**
 - This methodology is extensively used to obtain the prices of Fixed Income Securities.
- Conjecture (this comes with experience)

$$f(\nu, t; \tau) = \exp \left(a_0(t; \tau) + a_1(t; \tau)\nu + a_2(t; \tau)\nu^2 \right)$$

- Then we obtain

$$\begin{aligned} \frac{\partial f}{\partial t} &= (a'_0 + a'_1 \nu_t + a'_2 \nu_t^2) f \\ \frac{\partial f}{\partial \nu} &= (a_1 + 2a_2 \nu_t) f \\ \frac{\partial^2 f}{\partial \nu^2} &= (2a_2 + (a_1 + 2a_2 \nu_t)^2) f \end{aligned}$$

- Substitute and delete f on both sides, to find

$$r = a'_0 + a'_1\nu + a'_2\nu^2 + \left(\frac{1}{\gamma}(r - \rho) + \frac{\nu_t^2}{2\gamma} \left(\frac{1}{\gamma} - 1 \right) \right) + (a_1 + 2a_2\nu) \left(\hat{A}_0 + \hat{A}_1\nu + \frac{\nu_t}{\gamma} \hat{\sigma} \right) + \frac{1}{2}(2a_2 + (a_1 + 2a_2\nu)^2) \hat{\sigma}^2$$

- Finally, bunch up together terms in ν , ν^2 etc.
- One obtains:

$$0 = a'_0 - r + \left(\frac{1}{\gamma}(r - \rho) \right) + a_1 \hat{A}_0 + \frac{1}{2} (2a_2 + a_1^2) \hat{\sigma}^2 + \left(a'_1 + a_1 \left(\hat{A}_1 + \frac{\hat{\sigma}}{\gamma} \right) + 2a_2 \hat{A}_0 + 2a_1 a_2 \hat{\sigma}^2 \right) \nu_t + \left(a'_2 + \frac{1}{2\gamma} \left(\frac{1}{\gamma} - 1 \right) + 2a_2 \left(\hat{A}_1 + \frac{\hat{\sigma}}{\gamma} \right) + 2a_2^2 \hat{\sigma}^2 \right) \nu_t^2$$

- This equation is satisfied if the following system of ODEs is satisfied

$$a'_2 + 2a_2 \left(\hat{A}_1 + \frac{\hat{\sigma}}{\gamma} \right) + 2a_2^2 \hat{\sigma}^2 + \frac{1}{2\gamma} \left(\frac{1}{\gamma} - 1 \right) = 0$$

$$a'_1 + a_1 \left(\hat{A}_1 + \frac{\hat{\sigma}}{\gamma} \right) + 2a_2 \hat{A}_0 + 2a_1 a_2 \hat{\sigma}^2 = 0$$

$$a'_0 - r + \frac{1}{\gamma}(r - \rho) + a_1 \hat{A}_0 + \frac{1}{2} (2a_2 + a_1^2) \hat{\sigma}^2 = 0$$

- Note that the system can be easily solved recursively: Solve the first equation (for a_2), then plug in the solution into the second equation (for a_1) and then finally, obtain the solution of the last equation (for a_0)
- It is possible to find exact closed formulas for a_2 and a_1 . However, this is as easy to obtain numerically.
- ODEs are infinitely simpler than PDE, as you can solve them backward: Start with the final condition $a_0(\tau) = a_1(\tau) = a_2(\tau) = 0$ and then simply move backwards. Below are the details.
- Once we have the solution for $a_0(t; \tau)$, $a_1(t; \tau)$ and $a_2(t; \tau)$ for every $\tau \in [0, T]$, the function $F(\nu, t; T)$ can be obtained easily.
- In fact, recall that

$$\begin{aligned}
 V(C, \nu, t; T) &= CF(\nu, t; T) = E_t^Q \left[\int_t^T e^{-r(\tau-t)} C_\tau d\tau \right] \\
 &= \int_t^T E_t^Q \left[e^{-r(\tau-t)} C_\tau \right] d\tau = \int_t^T v(C, \nu, t; \tau) d\tau \\
 &= C \int_t^T f(\nu, t; \tau) d\tau
 \end{aligned}$$

- This implies that simply

$$F(\nu, t; T) = \int_t^T f(\nu, t; \tau) d\tau = \int_t^T \exp(a_0(t; \tau) + a_1(t; \tau)\nu_t + a_2(t; \tau)\nu_t^2) d\tau$$

- The portfolio holdings require the computation of

$$\frac{\partial F}{\partial \nu} = \int_t^T (a_1(t; \tau) + a_2(t; \tau)\nu_t) \exp(a_0(t; \tau) + a_1(t; \tau)\nu_t + a_2(t; \tau)\nu_t^2) d\tau$$

- To conclude, we have the following portfolio rule

$$\vartheta_t = \text{Myopic Demand} + \text{Hedging Demand}$$

with

$$\begin{aligned} \text{Myopic Demand} &= \frac{\mu_t - r}{\gamma \sigma^2} \\ \text{Hedging Demand} &= \left(\frac{1}{F} \frac{\partial F}{\partial \nu} \right) \frac{\tilde{\sigma}}{\sigma} \\ &= \frac{\tilde{\sigma} \int_t^T (a_1(t; \tau) + a_2(t; \tau)\nu_t) f(\nu_t, t; \tau) d\tau}{\sigma \int_t^T f(\nu_t, t; \tau) d\tau} \end{aligned}$$

- Also note that the optimal consumption was given by $C_t = W_t F(\nu, t; T)^{-1}$ implying that the C/W ratio is given by

$$\frac{C}{W} = \frac{1}{\int_t^T f(\nu_t, t; \tau) d\tau}$$

A Calibration

- An obvious application of the setting above is the one where returns μ_t are predictable from the dividend price ratio.
- We can think then of μ_t to be just

$$\mu_t = \alpha + \beta \log(D_t/P_t)$$

where α and β are the regression coefficients of some sort of predictive regression

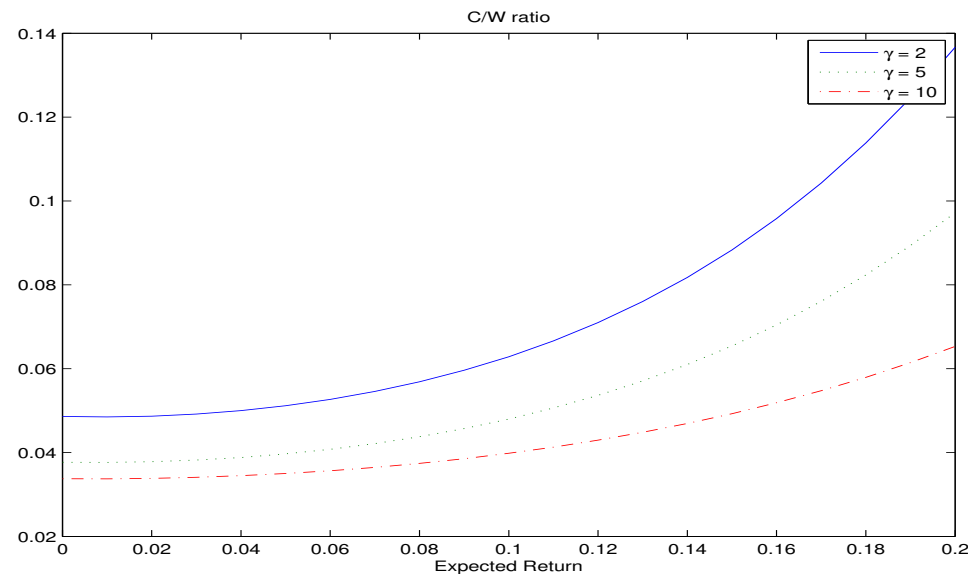
- Note that if $\log(D_t/P_t)$ follows a mean reverting process, so does μ_t and so we are back to the case discussed in this section.
- Also, there is a natural negative correlation between returns and D/P ratio: a negative return implies that P_t decreased. Since dividends do not move much, this implies that D/P went up.
- How do we impose a negative correlation in the model? Just assume that $\hat{\sigma} < 0$
- The following parameters have been used by many, including Barberis (JF 2000), and Wachter (JFQA 2002)

Parameter Choice

Rate of time preference ρ	0.0624
Risk free rate r	0.0168
Volatility of stock prices σ	0.1510
Volatility of ν_t	-0.0655
Mean Reversion A_1	-0.0226
Drift A_0	0.0062

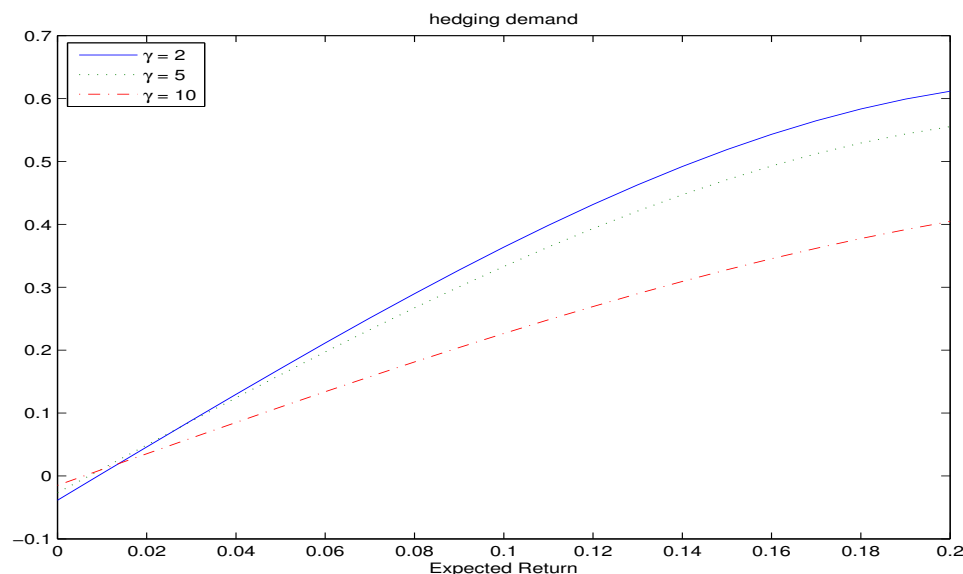
- The following figures show the C/W ratio, Hedging Demand and Total Demand as a function of expected return μ_t

Figure 1: C/W ratio



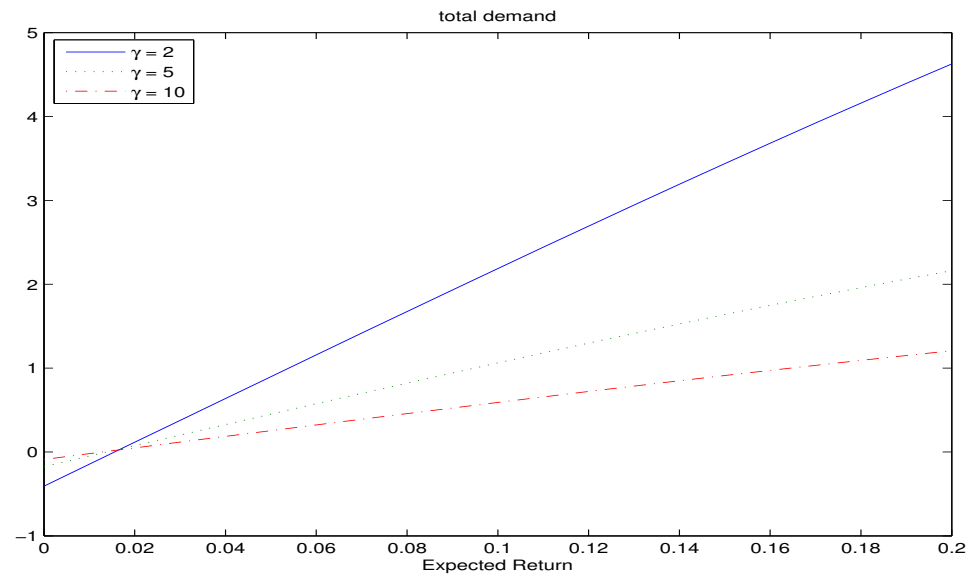
- Note that the C/W ratio declines with γ , as intuition would have it: Higher risk aversion implies a higher desire to save and thus consume less.

Figure 2: Hedging Demand



- The hedging demand is positive. The intuition is simple:
 - If we have a bad shock to returns, we have that μ_t increases (intuitively, the D/P increases, implying higher expected return).
 - But a higher μ_t implies that investor now want to buy more of the stock.
 - Anticipating this correlation, the investor buys more of the stock today, compared to the case where the hedging demand is zero.
- This finding is bad news for the portfolio holding puzzle: We already showed that the agent would hold too much of the stock even with simple myopic demand (no time varying investment opportunity set).
- The total demand now of the stock is even higher, deepening the puzzle.

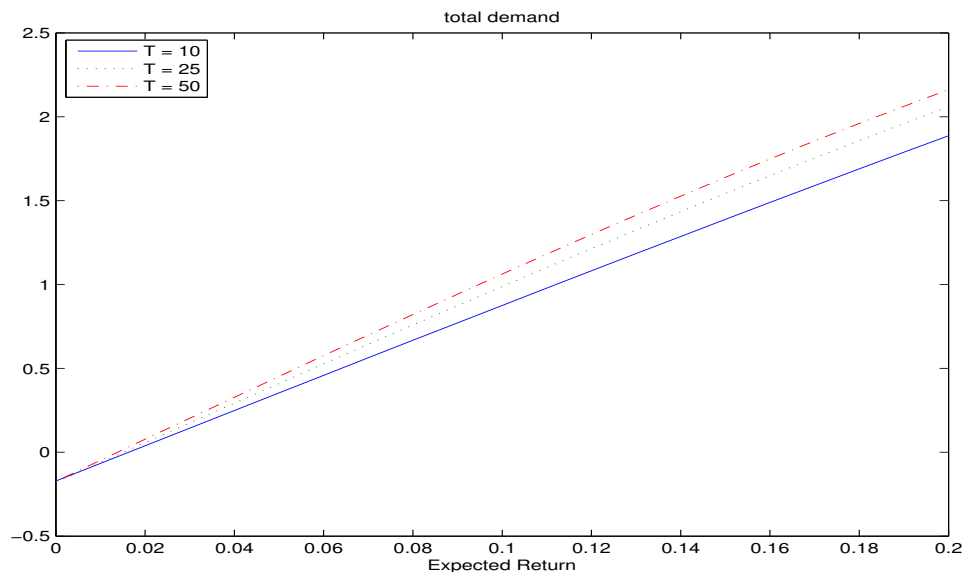
Figure 3: Total Demand



- We will see other channels that would decrease the holding of stocks later on.

- The predictability, however, helps a little to generate asset holding that depend on life cycle
- Using the same parameters, with $\gamma = 5$ but for three different maturities T we obtain the following.

Figure 4: Total Demand



- As it can be seen, the shorter the time to “death” the lower the share in stocks, especially if current expected return is high.
- In this case, mean reversion kicks in and the investor is wary about the negative consequences of a decrease in expected returns.