Teaching Notes #1
Review of Dynamic Equilibrium Models with Complete Markets[1]

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[1]These teaching notes draw heavily on Duffie (1996, Chapters 5 and 6) and Karatzas and Shevre (1999, Chapter 1) . They are intended for students of Business 35909 only. Please, do not distribute without my prior consent.

1 Introduction

- In these TNs I introduce the notation and some terminology.
- I won't explain all the details, and you can read TN#0 for the technical assumptions and so on.
- I will only expand on the necessary concepts to understand what is going on.
- Results from probability theory are recalled here and there. They are also all nicely collected in TN#0.
- In the first section, I introduce the canonical model of security prices. I will review some of the standard terminology by appealing to a simple, discrete time example.

2 A Model of Security Prices

- Fix a complete probability space $(\Omega, \mathcal{F}, P)$ and a time interval $[0, T]$.
  - Ω = set of states of nature [e.g. $\Omega = \{\omega_1, .., \omega_4\}$];
  - $\mathcal{F} = \sigma-$ *algebra* on Ω [e.g. $\mathcal{F} = 2^\Omega$ = set of all subset of Ω];
  - P = *Probability measure* on Ω [e.g. p =$\{p_1, ..., p_4\}$ such that $p_i \geq 0$ and $\sum_{i=1}^4 p_i = 1$].
  - *Complete* = technical requirement. It means that all the subsets of sets with zero probability are part of $\mathcal{F}$;
- Let $\{\mathcal{F}_t\}$ be a *filtration* on $(\Omega, \mathcal{F})$, that is, a family of sub-$\sigma-$algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$.

- Intuitively, the "filtration" describes the evolution of information over time (think about learning).
  - * Example: $\Omega = \{\omega_1, ..., \omega_4\}$. Then a filtration could be

$$\begin{aligned} \mathcal{F}_0 &= \{\emptyset, \Omega\} \\ \mathcal{F}_1 &= \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\} \\ \mathcal{F}_2 &= 2^{\Omega} \end{aligned}$$

  - * In this example, at time $t = 0$ there is no information, at time $t = 1$ we know whether $\{\omega_1, \omega_2\}$ realized or $\{\omega_3, \omega_4\}$ realized. At time $t = 2$ we have perfect information.
  - * This is the information evolution typical of a binomial tree. For instance, let

$$\begin{aligned} \omega_1 &= \{1, 1\}, \omega_2 = \{1, -1\} \\ \omega_3 &= \{-1, 1\}, \omega_4 = \{-1, -1\} \end{aligned}$$

  - * where $\{-1, 1\}$ are the possible realizations of a process $\{B_t\}$:

$t = 0$	$t = 1$	$t = 2$
		$B_2 = 1 \longrightarrow \omega_1$
	$B_1 = 1 \longrightarrow \{\omega_1, \omega_2\}$	$B_2 = -1 \longrightarrow \omega_2$
$B_0 = 0$		
	$B_1 = -1 \longrightarrow \{\omega_3, \omega_4\}$	$B_2 = 1 \longrightarrow \omega_3$
		$B_2 = -1 \longrightarrow \omega_4$

- The pair $\{B_t, \mathcal{F}_t\}$ has a special property: For every value $\widetilde{B}$ that B_t can take at t (that is, $\widetilde{B} = 1$ or -1), the set $\left\{\omega_i : B_t(\omega_i) = \widetilde{B}\right\} \in \mathcal{F}_t$.
  - E.g. at time $t = 1$, $\{\omega_i : B_1(\omega_i) = 1\} = \{\omega_1, \omega_2\} \in \mathcal{F}_1$, $\{\omega_i : B_1(\omega_i) = -1\} = \{\omega_3, \omega_4\} \in \mathcal{F}_1$
  - At time $t = 2$, $\{\omega_i : B_2(\omega_i) = 1\} = \{\omega_1, \omega_3\} \in \mathcal{F}_2$, and so on.
- B_t is said to be *measurable* with respect to $\mathcal{F}_t$.
- The process $\{B_t\}$ is said to be *adapted* to the filtration $\{\mathcal{F}_t\}$.
  - That is, given our information at time t described by $\mathcal{F}_t$, we can fully "observe" the value of B_t by observing a realization in $\mathcal{F}_t$.
  - If B_1 was instead such that $B_1(\omega_3) = -1$ and $B_1(\omega_4) = -2$, then by only observing the set $\{\omega_3, \omega_4\}$ we still cannot tell what is the value of B_1 if we move down. That is, we cannot "measure" B_1 given our information set $\mathcal{F}_2 = \{\omega_3, \omega_4\}$.
  - This has an additional implication: at time $t = 1$, suppose we only know that $P(\{\omega_1, \omega_2\}) = .3$ and $P(\{\omega_3, \omega_4\}) = .7$. If B_1 is not measurable $(B_1(\omega_3) \neq B_1(\omega_4))$ we won't be able to assign the probabilities to the actual realization of B_1 (that is, if we observe $B_1 = 2$, what was the probability of this event? We cannot tell, given the probability space).

- A filtration $\{\mathcal{F}_t\}$ is said to be *generated* by $\{B_t\}$ if it is exactly $\{B_t\}$ that reveals information (technically, $\mathcal{F}_t$ is the smallest $\sigma-$algebra that makes B_t measurable).
  - In the tree above, $\mathcal{F}_t$ was generated by $\{B_t\}$.
  - Another filtration that is not generated by B_t is, for instance,
$$\mathcal{F}_0 = 2^\Omega, \mathcal{F}_1 = 2^\Omega, \mathcal{F}_2 = 2^\Omega$$
  - That is, the agent knows everything from the start.

2.1 Trading Strategies

- Let $B = \{B_t, \mathcal{F}_t\}$ be a standard Brownian motion defined on this space with $\{\mathcal{F}_t\}$ being the filtration generated by $\{B_t\}$; that is:
  1. $B_0 = 0$;
  2. $B_t - B_s$ is independent of $\mathcal{F}_s$;
  3. $B_t - B_s \sim \mathcal{N}(0, t-s)$.
  - Technical requirement: the filtration $\mathcal{F}_t$ must be augmented by the $P-$null sets.
- A *trading strategy* $\theta = \{\theta_t\}$ is a process defined on $(\Omega, \mathcal{F}, P)$ that for every $\omega \in \Omega$ and every t specifies the number $\theta_t(\omega)$ of units of the security to hold.
- It is natural to require that $\{\theta_t\}$ be adapted to the filtration $\{\mathcal{F}_t\}$. We denote by $\mathcal{L}$ the set of processes adapted to $\{\mathcal{F}_t\}$.

2.2 Trading Gains

- Assume first that $\{B_t\}$ represents a price process and there are no dividends.
- Clearly, if $\theta_t = \overline{\theta} =$ constant for $t \in [t_1, t_2)$, the gain is

$$\text{Gain} = \overline{\theta}\,(B_{t_2} - B_{t_1})$$

- More generally, if the process $\theta \in \mathcal{L}$ is *simple*, i.e. piecewise constant on a partition $[t_0, .., t_n]$ with $t_n = t$, we have

$$\text{Gain}_t = \sum_{i=0}^{n-1} \theta_{t_i}\left(B_{t_{i+1}} - B_{t_i}\right)$$

- As we increase the number of times rebalancing is allowed in the interval $[0, t]$, we obtain the convergence (in probability) of the sum above to the *stochastic integral*

$$\text{Gain}_t = \int_0^t \theta_s dB_s$$

- We shall assume that $\theta \in \mathcal{L}^2 = \left\{\theta \in \mathcal{L} : \int_0^T |\theta_t|^2 dt < \infty \text{ a.s.}\right\}$.
- **Result 1**: for any bounded $\theta \in \mathcal{L}^2$, the stochastic integral $\int_0^t \theta_s dB_s$ is a martingale.
- This is unduly restrictive, and it turns out that $\int_0^t \theta_s dB_s$ under weaker assumptions. However, in these TNs I will retrict the attention only to bounded trading strategies for simplicity (and ease of notation).

- **Definition:** An adapted, integrable process X_t is
  - a *martingale* if $E\left[X_t|\mathcal{F}_s\right] = X_s$ for $t \geq s$;
  - a *sub-martingale* if $E\left[X_t|\mathcal{F}_s\right] \geq X_s$ for $t \geq s$;
  - a *super-martingale* if $E\left[X_t|\mathcal{F}_s\right] \leq X_s$ for $t \geq s$;

- To generalize the model, let's introduce Ito processes to describe the dynamics of security prices.

- Ito processes are of the form

$$S_t = x + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s \tag{1}$$

- where:
  - x is a real number (the initial condition);
  - $\mu_s \in \mathcal{L}^1 = \left\{\theta \in \mathcal{L} : \int_0^T |\theta_t| dt < \infty \text{ a.s.}\right\}$;
  - $\sigma_s \in \mathcal{L}^2 = \left\{\theta \in \mathcal{L} : \int_0^T |\theta_t|^2 dt < \infty \text{ a.s.}\right\}$.

- We can write (1) in its "differential form" as a short-hand

$$dS_t = \mu_t dt + \sigma_t dB_t \tag{2}$$

- We have

$$\begin{aligned} \frac{dE_t\left[S_\tau\right]}{d\tau}|_{\tau=t} &= \mu_t \text{ a.s.} \\ \frac{dVar_t\left[S_\tau\right]}{d\tau}|_{\tau=t} &= \sigma_t^2 \text{ a.s.} \end{aligned}$$

- Hence, μ_t is the conditional expected rate of change of S at time t and σ_t^2 is the rate of change of the conditional variance of S at time t.
- With abuse of notation, we will often write "$E_t[dS_t] = \mu_t dt$" and "$Var_t[dS_t] = \sigma_t^2 dt$".
- μ_t is called "drift" and σ_t is called "diffusion" term of (2).
- Given an Ito process S as in (1), let

$$\mathcal{L}(S) = \left\{\theta \in \mathcal{L} : \theta \cdot \mu \in \mathcal{L}^1 \text{ and } \theta \cdot \sigma \in \mathcal{L}^2\right\}$$

- For a given (adapted) trading strategy $\theta \in \mathcal{L}(S)$, the trading gain between 0 and t is

$$G_t \equiv \int_0^t \theta_s dS_t = \int_0^t \theta_s \mu_s ds + \int_0^t \theta_s \sigma_s dB_s$$

- In a multidimensional set up, let $\mathbf{B} = \left(B^1, ..., B^d\right)$ be a $d \times 1$ Brownian motion in $\mathcal{R}^d$ and let $\{\mathcal{F}_t\}$ be the filtration generated by it.
- Let $\mathbf{S} = \left(S^1, .., S^N\right)$ denote the price of N securities whose dynamics is described by the Ito process

$$\mathbf{S}_t = \mathbf{x} + \int_0^t \boldsymbol{\mu}_s ds + \int_0^t \boldsymbol{\sigma}_s d\mathbf{B}_s \tag{3}$$

- where:
  - $\mathbf{x}$ is a N dimensional real vector;

  - μ is a N dimensional vector of $\mathcal{L}^1$ processes;
  - $\boldsymbol{\sigma}$ is an $N \times d$ matrix, such that $\sigma^{ij} \in \mathcal{L}^2$.

- An $N-$dimensional trading strategy $\boldsymbol{\theta} = \left(\theta^1, ..., \theta^N\right)$ is a $1 \times N$ vector of adapted processes θ^i.

- We assume

$$\boldsymbol{\theta} \in \mathcal{L}\left(\mathcal{S}\right) = \left\{ \begin{array}{c} \boldsymbol{\theta} \in \mathcal{L} : \boldsymbol{\theta} \cdot \boldsymbol{\mu} \in \mathcal{L}^1 \text{ and } \boldsymbol{\theta} \cdot \boldsymbol{\sigma}^i \in \mathcal{L}^2, \\ \text{where } \boldsymbol{\sigma}^i \text{ is the } i-th \text{ column of } \boldsymbol{\sigma} \end{array} \right\}$$

- Hence, the trading gain process is given by the stochastic integral

$$G_t = \int_0^t \boldsymbol{\theta}_\tau d\mathbf{S}_\tau = \int_0^t \boldsymbol{\theta}_\tau d\boldsymbol{\mu}_\tau + \int_0^t \boldsymbol{\theta}_\tau \cdot \boldsymbol{\sigma}_\tau d\mathbf{B}_\tau$$

3 Arbitrage

- A trading strategy $\boldsymbol{\theta} = \left(\theta^1, ..., \theta^N\right)$ is *self financing* if the value of the position at time t equals the value of the initial position plus any trading gains:

$$\begin{aligned} \boldsymbol{\theta}_t \cdot \mathbf{S}_t &= \boldsymbol{\theta}_0 \cdot \mathbf{S}_0 + G_t \\ &= \boldsymbol{\theta}_0 \cdot \mathbf{S}_0 + \int_0^t \boldsymbol{\theta}_\tau d\mathbf{S}_\tau \end{aligned}$$

- Let the *short rate process* be given by an adapted process $r \in \mathcal{L}^1$ and define the bond price process by

$$\beta_t = \beta_0 \exp\left(\int_0^t r_s ds\right)$$

- We often identify the β process with the security process S_0.
- The trading strategy $\boldsymbol{\theta} = \left(\theta^1, ..., \theta^N\right)$ is an *arbitrage* if
  1. It is self financing; and
  2. $\boldsymbol{\theta}_0 \cdot \mathbf{S}_0 < 0 \Longrightarrow \boldsymbol{\theta}_T \cdot \mathbf{S}_T \geq 0$ or $\boldsymbol{\theta}_0 \cdot \mathbf{S}_0 \leq 0 \Longrightarrow \boldsymbol{\theta}_T \cdot \mathbf{S}_T > 0$.
- A financial market that admits no arbitrage is termed *viable*.
- The question is "what properties should $\mathbf{S}$ to make it a viable financial market?"

3.1 Change of Numeraire and State Price Deflators

- It is often convenient to change numerarie to exploit some properties of stochastic processes;
- Let $\mathbf{S}$ be given and consider a strictly positive Ito process Y, which we call *deflator*.
- Let

$$dY_t = \mu_{Y,t} dt + \boldsymbol{\sigma}_{Y,t} d\mathbf{B}_t$$

- Let's define

$$\mathbf{S}_t^Y = \mathbf{S}_t Y_t$$

- *Numeraire Invariance Theorem*: Suppose Y is a deflator. Then, a trading strategy $\boldsymbol{\theta}$ is self-financing with respect to $\mathbf{S}$ if and only if it is self-financing with respect to $\mathbf{S}^Y$
  - The theorem is intuitively obvious: the properties of a trading strategy cannot depend on whether I express quantities in apples, dollars or widgets.

– Formally, the result is immediate from Ito's Lemma, which states the following:

- **Result:** *Ito's Lemma*: Consider the Ito' process

$$d\mathbf{X}_t = \boldsymbol{\theta}_t dt + \boldsymbol{\Sigma}_t d\mathbf{B}_t \tag{4}$$

– and let $g : [0, T] \times \mathcal{R}^d \longrightarrow \mathcal{R}^p$ be a twice continuously differentiable function. Then, the $p-$dimensional vector $\mathbf{Y}_t = g(t, \mathbf{X}_t)$ is an Ito's process, whose component number k is given by

$$\begin{aligned} dY_t^k &= \frac{\partial g_k}{\partial t}(t, \mathbf{X}_t)\, dt + \sum_{i=1}^n \frac{\partial g_k}{\partial X^i}(t, \mathbf{X}_t)\, dX_t^i \\ &\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 g_k}{\partial X^i \partial X^j}(t, \mathbf{X}_t) \left(dX_t^i\right) \left(dX_t^j\right) \end{aligned}$$

- To show the numeraire invariance theorem: Let $W_t = \boldsymbol{\theta}_t \cdot \mathbf{S}_t = \boldsymbol{\theta}_0 \mathbf{S}_0 + G_t = \boldsymbol{\theta}_0 \mathbf{S}_0 + \int_0^t \boldsymbol{\theta}_\tau d\mathbf{S}_\tau$. Let $W_t^Y = W_t Y_t$. From Ito's Lemma

$$\begin{aligned} dW_t^Y &= Y_t dW_t + W_t dY_t + \sigma_W \sigma_Y' dt \\ &= Y_t \boldsymbol{\theta}_t d\mathbf{S}_t + \boldsymbol{\theta}_t \cdot \mathbf{S}_t dY_t + \left(\boldsymbol{\theta}_t \cdot \boldsymbol{\sigma}_{S,t}\right) \sigma_{Y,t}' dt \\ &= \boldsymbol{\theta}_t \left(Y_t d\mathbf{S}_t + \mathbf{S}_t dY_t + \boldsymbol{\sigma}_{S,t} \sigma_{Y,t}' dt\right) \\ &= \boldsymbol{\theta}_t d\mathbf{S}_t^Y \end{aligned}$$

- Hence, $W_t^Y = \boldsymbol{\theta}_t \cdot \mathbf{S}_t^Y = \boldsymbol{\theta}_0 \mathbf{S}_0^Y + \int_0^t \boldsymbol{\theta}_\tau d\mathbf{S}_\tau^Y$ if and only if $W_t = \boldsymbol{\theta}_t \cdot \mathbf{S}_t = \boldsymbol{\theta}_0 \mathbf{S}_0 + \int_0^t \boldsymbol{\theta}_\tau d\mathbf{S}_\tau$. $\square$

- **Corollary:** An immediate implication is that if Y is deflator, then a trading strategy is an arbitrage with respect to $\mathbf{S}$ if and only if it is an arbitrage with respect to $\mathbf{S}^Y$.

- **The point is "what is a good deflator"?**
- A *state-price deflator* is a deflator π with the property that the deflated price processes $\mathbf{S}^\pi$ are martingales.
- The key result will be that there exists a state-price deflator if and only if there is no arbitrage.

- Before we explore this, we need some restrictions on the admissible trading strategies or wealth processes.
  - **Example: Doubling Strategies.**
  - Let B be a uni-dimensional BM.
  - Let $\mathbf{S} = (\beta, S)$ be the price processes where $S_0 = 1$; $\beta_t = 1$ for all t and

$$dS_t = S_t dB_t$$

  - Notice that $\mathbf{S}$ is a martingale. However, we can construct an arbitrage as follows.
  - Let $\alpha > 0$ and consider the process φ given by $\varphi_0 = 0$ and

$$d\varphi_t = \frac{1}{\sqrt{T-t}} dB_t$$

– Finally consider the stopping time

$$\tau = \inf \{t : \varphi_t = \alpha\}$$

– Since the volatility of φ explodes to infinity as t approaches to T, it can be shown that almost surely $0 < \tau < T$.

– Consider now the trading strategy $\boldsymbol{\theta} = (a, b)$ given by

$$\begin{aligned} \text{Stock} &: \quad b_t = \frac{1_{\{t \leq \tau\}}}{S_t \sqrt{T - t}} \\ \text{Bonds} &: \quad a_t = -b_t S_t + \int_0^t b_u dS_u \end{aligned}$$

– Notice that $\boldsymbol{\theta}_0 \cdot \mathbf{S}_0 = 0$ and that

$$\begin{aligned} \boldsymbol{\theta}_t \cdot \mathbf{S}_t &= -b_t S_t + \int_0^t b_u dS_u + b_t S_t \\ &= \int_0^t b_u dS_u \end{aligned}$$

– Hence, it is self-financing.

– Finally, it is also clear that since $\tau < T$ almost surely, we have

$$\boldsymbol{\theta}_\tau \cdot \mathbf{S}_\tau = \int_0^\tau b_u dS_u = \int_0^\tau 1_{\{t \leq \tau\}} \frac{1}{\sqrt{T - u}} dB_u = \alpha > 0$$

– Hence, it is an arbitrage.

- Why do we have an arbitrage? This is an artifact of the continuous time nature of the exercise that let you bet infinitely frequently in any interval of time.

- In the literature there are two ways to eliminate doubling strategies:

  1. **Integrability condition**: Given a state-price deflator π (i.e. such that $\mathbf{S}^\pi$ is a martingale) the only admissible strategies are those $\boldsymbol{\theta} \in \mathcal{H}^2(S^\pi)$, where

$$\mathcal{H}^2(S^\pi) = \left\{ \begin{array}{c} \boldsymbol{\theta} \in \mathcal{L} : E\left[\left(\int_0^T \boldsymbol{\theta}_t \cdot \boldsymbol{\mu}_t^\pi dt\right)^2\right] < \infty \text{ and for all } i \\ E\left(\int_0^T \left(\boldsymbol{\theta}_t \cdot \boldsymbol{\sigma}_t^i\right)^2 dt\right) < \infty \end{array} \right\}$$

  2. **Credit constraints**: There exists a (negative) constant k such that $\boldsymbol{\theta}_t \cdot \mathbf{S}_t^\pi \geq k$ almost surely. Let $\underline{\Theta}(\mathbf{S}^\pi)$ denote the set of strategies satisfying this bounds.

- The previous strategy does not satisfy either of these conditions.

- We have the following:

Proposition 1: For any state-price deflator π, there is no arbitrage if (a) θ is bounded, or (b) $\theta \in \mathcal{H}^2(\mathbf{S}^\pi)$ or $\underline{\Theta}(\mathbf{S}^\pi)$.

- It is instructive to go through the proof (of the first simple claim. The second is slightly more involved. See Duffie's book):

- (a) Suppose $\boldsymbol{\theta}$ is a self-financing bounded strategy. Since $\mathbf{S}^\pi$ is a martingale, Result 1 above implies that also $\int_0^T \boldsymbol{\theta}_t d\mathbf{S}_t^\pi$ is a martingale and hence $E\left[\int_0^T \boldsymbol{\theta}_t d\mathbf{S}_t^\pi\right] = 0$. Since $\boldsymbol{\theta}$ must be self-financing with respect to $\mathbf{S}^\pi$ as well, that is

$$\boldsymbol{\theta}_T \cdot \mathbf{S}_T^\pi = \boldsymbol{\theta}_0 \cdot \mathbf{S}_0^\pi + \int_0^T \boldsymbol{\theta}_\tau d\mathbf{S}_\tau^\pi$$

we have

$$\boldsymbol{\theta}_0 \mathbf{S}_0^\pi = E\left[\boldsymbol{\theta}_T \cdot \mathbf{S}_T^\pi\right]$$

- Hence, $\boldsymbol{\theta}_T \cdot \mathbf{S}_T^\pi \geq 0$ implies $\boldsymbol{\theta}_0 \cdot \mathbf{S}_0^\pi \geq 0$ and likewise $\boldsymbol{\theta}_T \cdot \mathbf{S}_T^\pi > 0$ implies $\boldsymbol{\theta}_0 \mathbf{S}_0^\pi > 0$.
- Therefore, $\boldsymbol{\theta}$ cannot be an arbitrage for $\mathbf{S}^\pi$ and hence neither for $\mathbf{S}$. $\square$

3.2 No Arbitrage and Equivalent Martingale Measures

- Given a measurable space $(\Omega, \mathcal{F})$, a probability measure Q is *equivalent* to P, if for every $A \in \mathcal{F}$, $Q(A) = 0$ if and only if $P(A) = 0$.
- A probability measure Q is an *equivalent martingale measure* for the price process $\mathbf{S}$ if
  1. $\mathbf{S}$ is a martingale with respect to Q;
  2. The Radon-Nikodym derivative dQ/dP has finite variance.
- **Result**: *Radon-Nikodym theorem:* Let Q and P be two measures with Q absolutely continuous with respect to P.

Then there exists a *non-negative* random variable ξ with $E^P[\xi] = 1$ such that

$$Q(A) = \int_A \xi(\omega)\, dP(\omega)$$

for all $A \in \mathcal{F}$. In addition, Q and P are equivalent if and only if ξ is strictly positive. ξ is called Radon Nikodym derivative, and it is often denoted by dQ/dP.

- Two properties: Let Z be a random variable such that $E^Q[|Z|] < \infty$. Then

  1. Change of measure: $E^Q[Z] = E^P[\xi Z]$
  2. Bayes theorem: If $\mathcal{G} \subset \mathcal{F}$,

  $$E^Q[Z|\mathcal{G}] = \frac{E^P[\xi Z|\mathcal{G}]}{E^P[\xi|\mathcal{G}]}$$

- Examples:

  1. $\Omega = \{\omega_1, .., \omega_n\}$, $p = \{p_1, ..., p_n\}$, $q = \{q_1, .., q_n\}$.
     - Assume $p_i > 0$ and $q_i > 0$. Define the Radon Nikodym derivative $\xi = \{\xi_1, .., \xi_n\}$ by $\xi_i = q_i/p_i$.
     - It is a random variable. In addition, the properties above are obviously satisfied:
     - First, $E^P[\xi_i] = \sum_i p_i \xi_i = \sum_i q_i = 1$.
     - Second, we obtain that for every $A = \{\omega_j\} \in 2^\Omega$, $Q(A) = \sum_j q_j = \sum_j \xi_j p_j$.
     - Third, given a random variable $z = \{z_1, .., z_n\}$ we also have $\mathrm{E}^Q[z] = \sum_i q_i z_i = \sum_i p_i \xi_i z_i = E^P[\xi z]$.

– Finally, suppose we learn that either ω_1 or ω_2 is the true state, with $\widetilde{q}_1 = q_1/(q_1+q_2)$. Then

$$\begin{aligned} E^Q[z|\mathcal{G}] &= \widetilde{q}_1 z_1 + \widetilde{q}_2 z_2 = \frac{q_1}{q_1+q_2} z_1 + \frac{q_2}{q_1+q_2} z_2 \\ &= \frac{p_1\xi_1 z_1 + p_2\xi_2 z_2}{p_1\xi_1 + p_2\xi_2} = \frac{p_1+p_2}{p_1+p_2}\frac{p_1\xi_1 z_1 + p_2\xi_2 z_2}{p_1\xi_1 + p_2\xi_2} \\ &= \frac{\widetilde{p}_1\xi_1 z_1 + \widetilde{p}_2\xi_2 z_2}{\widetilde{p}_1\xi_1 + \widetilde{p}_2\xi_2} = \frac{E^P[\xi z|\mathcal{G}]}{E^P[\xi|G]} \end{aligned}$$

2. An example of a Radon-Nikodym derivative that will be used often is

$$\frac{dQ^\theta}{dP} = \xi_t^\theta = \exp\left(-\int_0^t \boldsymbol{\theta}_s d\mathbf{B}_s - \frac{1}{2}\int_0^t \boldsymbol{\theta}_s \cdot \boldsymbol{\theta}_s' ds\right)$$

where $\boldsymbol{\theta} = \left(\theta^1, ..., \theta^d\right) \in \left(\mathcal{L}^2\right)^d$ satisfies the condition

$$E\left[\exp\left(\frac{1}{2}\int_0^T \boldsymbol{\theta}_s \cdot \boldsymbol{\theta}_s' ds\right)\right] < \infty$$

– If this condition (*Novikov's condition*) is satisfied, it can be shown that ξ_t^θ is a $P-$martingale. Indeed, notice that by Ito's lemma we have[2]

$$d\xi_t^\theta = -\xi_t^\theta \boldsymbol{\theta}_s d\mathbf{B}_s$$

– In addition, since $\xi_0^\theta = 1$ we have

$$E^P\left[\xi_T^\theta\right] = 1$$

[2] To see it, consider the transformation $x_t = \log\left(\xi_t^\theta\right)$ first.

  - and thus satisfies the condition necessary to define a Radon-Nikodim derivative.

- We say that a price process $\mathbf{S}$ *admits* an equivalent martingale measure, if such a Q exists.

- A similar proof the one of proposition 1 also yields the following:

- **Proposition 2**: If the price process $\mathbf{S}$ admits an equivalent martingale measure, then there is no arbitrage with bounded θ, or in $\mathcal{H}^2(\mathbf{S})$ or $\underline{\Theta}^2(\mathbf{S})$.

  - *Proof*: Let Q be an equivalent martingale measure and let $\boldsymbol{\theta}$ be a self financing trading strategy.
  - (a) If $\boldsymbol{\theta}$ is bounded, because $\mathbf{S}$ is a martingale under Q and hence $E^Q\left[\int_0^T \boldsymbol{\theta}_t d\mathbf{S}_t\right] = 0$, we have immediately that

$$\boldsymbol{\theta}_0 \mathbf{S}_0 = E^Q\left[\boldsymbol{\theta}_T \cdot \mathbf{S}_T\right]$$

  - This implies that $\boldsymbol{\theta}$ cannot be an arbitrage (see argument in Proposition 1).
  - (b) and (c) are similar, but with more steps. □

4 The Market Price of Risk

- Let $\mathbf{S}$ be a price (Ito) process. We can write it as

$$d\mathbf{S}_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{B}_t$$

- Consider the system of N linear equations

$$\boldsymbol{\sigma}_t \cdot \boldsymbol{\eta}_t = \boldsymbol{\mu}_t \tag{5}$$

- where $\boldsymbol{\eta}_t$ is a $d \times 1$ vector of unknowns.
- From linear algebra, we know that this system admits 0, 1, or infinite solutions.
- If a solution to (5) exists, the process $\mathbf{S}$ is called *reducible.*
- Any $d\times 1$ vector satisfying (5) is giving a relationship between the expected return on each security μ_i and the risk stemming from the d Brownian motions $\left(\boldsymbol{\sigma}^{i1}, ..., \boldsymbol{\sigma}^{id}\right)$.
- $\boldsymbol{\eta}_t$ is called the "market price of risk."
- Often this is actually referred to a renormalized process.
- Let r_t be an interest rate process and $\beta_t = \exp\left(\int_0^t r_u du\right)$.
- Define the deflated process $\mathbf{S}_t^\beta = \mathbf{S}_t/\beta_t = \mathbf{S}_t \exp\left(-\int_0^t r_u du\right)$.
- Interestingly, we have

- **Result**: Suppose that for some deflator Y, $\mathbf{S}^Y$ is not reducible. Then, there are arbitrages even with bounded strategies, or, more generally, in both $\underline{\Theta}\left(\mathbf{S}^Y\right)$ and $\mathcal{H}^2\left(\mathbf{S}^Y\right)$. Moreover, there is no equivalent martingale measure for $\mathbf{S}^Y$.
  - Proof: See Duffie, Chapter 6.
- This is rather intuitive. The system (5) admits no solutions in those pathological cases such as the case where $d = 1$ (i.e. one Brownian motion), $N = 2$, $\sigma^1 > \sigma^2$ but $\mu^1 = \mu^2 = \mu$.
  - That is, the system (5) is

$$\begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \times \eta = \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

  - One can then set up a strategy with zero exposure to Brownian risk and positive return.
- If the system (5) admits multiple solutions (that is, $\text{rank}(\boldsymbol{\sigma}_t) < d$), we can single out a particular solution by defining

$$\eta_t = \widehat{\boldsymbol{\sigma}}_t' \left(\widehat{\boldsymbol{\sigma}}_t \widehat{\boldsymbol{\sigma}}_t'\right)^{-1} \widehat{\boldsymbol{\mu}}_t$$

- where $\widehat{\boldsymbol{\sigma}}_t$ is obtained by deleting the linearly dependent rows and $\widehat{\boldsymbol{\mu}}_t$ by deleting the corresponding elements (this is done ω by ω)
- Let now

$$\xi(\mathbf{S}) = \exp\left(-\int_0^T \boldsymbol{\eta}_t d\mathbf{B}_t - \frac{1}{2}\int_0^T \boldsymbol{\eta}_t \cdot \boldsymbol{\eta}_t' dt\right) \qquad (6)$$

- We say that $\mathbf{S}$ is L^2-*reducible* if $\mathbf{S}$ is reducible, $\exp\left(\frac{1}{2}\int_0^T \boldsymbol{\eta}_t \cdot \boldsymbol{\eta}_t' dt\right)$ has finite expectation and $\xi(\mathbf{S})$ has finite variance.
- We then have the result:
- **Theorem:** If $\mathbf{S}$ is L^2-reducible, then there exists an equivalent martingale measure for $\mathbf{S}$. Hence, there is no arbitrage.
- The result is a direct consequence of *Girsanov's Theorem*:
- **Result**: *Girsanov's Theorem:* Let $\theta \in \left(\mathcal{L}^2\right)^d$ be given and let

$$\xi_t^\theta = \exp\left(-\int_0^t \boldsymbol{\theta}_s d\mathbf{B}_s - \frac{1}{2}\int_0^T \boldsymbol{\theta}_s \boldsymbol{\theta}_s' ds\right)$$

be a martingale (Novikov condition suffices). Then a standard Brownian motion that is a martingale under Q^θ is defined by

$$\mathbf{B}_t^\theta = \mathbf{B}_t + \int_0^t \boldsymbol{\theta}_s ds \tag{7}$$

- This implies immediately:
- **Corollary 1**: Let $\mathbf{X}$ be an Ito process in $\mathcal{R}^N$:

$$\mathbf{X}_t = \mathbf{x} + \int_0^t \boldsymbol{\mu}_s ds + \int_0^t \boldsymbol{\sigma}_s d\mathbf{B}_s \tag{8}$$

where $\boldsymbol{\sigma}_s$ is $N \times d$.

- Suppose $\boldsymbol{\nu} = \left(\nu^1, ..., \nu^N\right)$ is a vector process in $\mathcal{L}^1$ such that there exists some $\boldsymbol{\theta} \in \left(\mathcal{L}^2\right)^d$ such that

$$\boldsymbol{\sigma}_t \boldsymbol{\theta}_t = \boldsymbol{\mu}_t - \boldsymbol{\nu}_t \tag{9}$$

- Then, if ξ^{θ} is a martingale, $\mathbf{X}$ is also an Ito process with respect to the probability space $\left(\Omega, \mathcal{F}, Q^{\theta}\right)$ and

$$\mathbf{X}_t = \mathbf{x} + \int_0^t \boldsymbol{\nu}_s ds + \int_0^t \boldsymbol{\sigma}_s d\mathbf{B}_s^{\theta}$$

- (to see this, just stick $d\mathbf{B}_s = d\mathbf{B}_s^{\theta} - \boldsymbol{\theta}_s dt$ into (8) and use (9))
- That is to say, the Girsanov's Theorem gives us a way to adjust probability assessments so that a given process can be rewritten as an Ito process with almost arbitrary drift.
- We can directly apply Girsanov's theorem to prove the theorem: If $\mathbf{S}$ is L^2-reducible, then Girsanov's theorem shows that Q is an equivalent martingale measure when defined by

$$dQ/dP = \xi\left(\mathbf{S}\right)$$

  – The result follows from Proposition 2. □

- Hence, reducibility is necessary and sufficient (when coupled with integrability conditions) for the absence of arbitrage.
- Notice that if $\text{rank}(\boldsymbol{\sigma}_t) = d$, then there is at most one solution to the system (5).
- It follows that there is at most one equivalent martingale measure.

  - If there were more than one, for each one of them we should have

$$dQ/dP = \xi(\mathbf{S})$$

  - where $\xi(\mathbf{S})$ is given by (6) (see TN#0, Corollary 2 to Girsanov's Theorem)

- This is also the case in which the market is dynamically complete, as we shall see.

5 State Prices and Equivalent Martingale Measure

- What is the relationship between the equivalent martingale measure Q and the state price deflators π?
- It turns out that they are basically the same thing.
- Let the $N \times 1$ vector process $\mathbf{S}$ be given and let r_t be the short rate process.
- Let $\mathbf{S}^\beta$ be the deflated process, where $\beta_t = \exp\left(\int_0^t r_u du\right)$
- Suppose that Q is an equivalent martingale measure for $\mathbf{S}^\beta$.
- Define the *density process* for Q by

$$\xi_t = E\left[\frac{dQ}{dP}|\mathcal{F}_t\right] \tag{10}$$

- where dQ/dP is the Radon-Nikodym derivative of Q with respect to P.

- From the properties of the Radon-Nikodym derivative, we then have that for every time t and $s > t$, any $\mathcal{F}_s$ measurable random variable W with $E^Q[|W|] < \infty$,

$$E^Q[W|\mathcal{F}_t] = \frac{E^P[\xi_s W|\mathcal{F}_t]}{E^P[\xi_s|\mathcal{F}_t]} = \frac{E^P[\xi_s W|\mathcal{F}_t]}{\xi_t} \tag{11}$$

- A state price deflator turns out to be

$$\pi_t = \beta_t^{-1}\xi_t \tag{12}$$

  - In fact, from (11), for every t and $u > t$ we have

$$\begin{aligned} E_t[\pi_u \mathbf{S}_u] &= E_t\left[\xi_u \beta_t^{-1}\mathbf{S}_u\right] = E_t\left[\xi_u \mathbf{S}_u^\beta\right] \\ &= \xi_t E_t^Q\left[\mathbf{S}_u^\beta\right] = \xi_t \mathbf{S}_t^\beta = \pi_t \mathbf{S}_t \end{aligned}$$

  - where the third equality stems from (11) and the fourth equality stems from the fact that Q is an equivalent martingale measure for $\mathbf{S}^\beta$.
  - Hence, $\mathbf{S}^\pi = \pi\mathbf{S}$ is a martingale, and hence π defined by (12) is a state-price deflator.

- Since all the equality and statements are "if and only if," we find an equivalence between state-price densities and equivalent martingale measures.

6 Consumption and Portfolio Selection in Complete Markets

- Markets are *complete* if any random variable Y with *finite variance* can be obtained as the final value $\boldsymbol{\theta}_T \cdot \mathbf{S}_T$ of a self-financing trading strategy.

- We have the following:

- **Proposition 3**: Suppose that there exists an equivalent martingale measure for the deflated price $\mathbf{S}/\beta$. Then, markets are complete if and only if $rank\,(\sigma) = d$ almost everywhere.[3]

- The proof of this proposition is instructive, again to illustrate the use of Girsanov's theorem, change of measures and the like. It is provided in the appendix.

- Define a *redundant security* given $(\beta, \mathbf{S})$ a security with price process Y such that there exists a self-financing strategy $\left(\theta^0, \boldsymbol{\theta}\right) \in \mathcal{H}^2\,(\beta, \mathbf{S})$ with terminal value $\left(\theta^0_T, \boldsymbol{\theta}_T\right) \cdot (\beta_T, \mathbf{S}_T) = Y_T$.

- Clearly, complete markets imply that every security is redundant.

[3] Two processes $\{a_t\}$ and $\{b_t\}$ are said to be equal almost everywhere if

$$E\left[\int_0^T |a_t - b_t| dt\right] = 0.$$

7 Optimal Consumption and Portfolio Selection

- We start from the "classic" Merton's problem.
- As usual, let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which a $d-$dimensional Brownian motion $\mathbf{B}$ is defined, along with its augmented filtration $\{\mathcal{F}_t\}$.
- Let $\mathbf{S}$ be a $d-$dimensional Ito-process

$$d\mathbf{S}_t = \mathbf{I}_S \boldsymbol{\mu} dt + \mathbf{I}_S \boldsymbol{\sigma} d\mathbf{B}_t$$

- where
  - $\mathbf{I}_S$ is the $d \times d$ diagonal matrix with S_t^i on the $ii-$th element;
  - $\boldsymbol{\mu}$ is a $d-$dimensional constant vector;
  - $\boldsymbol{\sigma}$ is a $d \times d$ constant matrix. Assume that $\boldsymbol{\sigma}$ is invertible.
- Hence, markets are complete.
- Assume there exists a constant interest rate r and let the bond price be given by

$$d\beta_t = r\beta_t dt \text{ with } \beta_0 > 0$$

- Consider a "small" investor endowed with a utility function $U(c, Z)$ defined on a stream of consumption $c = \{c_t\}_0^T \in \mathcal{L}^1$

and final wealth Z, assumed to be a $\mathcal{F}_T-$measurable non-negative random variable

$$U(c, Z) = E\left[\int_0^T u(c_t, t)\,dt + F(Z)\right] \tag{13}$$

- Assume:

  1. $F : R_+ \longrightarrow R$ is increasing and concave;
  2. $u : R_+ \times [0, T] \longrightarrow R$ is continuous, and for each t, it is increasing and concave in c_t with $u(0, t) = 0$;
  3. Either is strictly concave (or both).

- A trading strategy is $\left(\theta^0, \boldsymbol{\theta}\right) \in \boldsymbol{\mathcal{L}}(\beta, \mathbf{S})$.

- For ease of notation, let $\widetilde{\boldsymbol{\theta}} = \left(\theta^0, \boldsymbol{\theta}\right)$ and $\widetilde{\mathbf{S}} = (\beta, \mathbf{S})$.

- Let $w > 0$ be the initial wealth.

- A consumption plan (c, Z) and a trading strategy $\widetilde{\boldsymbol{\theta}}$ is budget feasible, denoted $\left(c, Z, \widetilde{\boldsymbol{\theta}}\right) \in \boldsymbol{\Lambda}(w)$ if they satisfy the dynamic budget constraint

$$\widetilde{\boldsymbol{\theta}}_t \cdot \widetilde{\mathbf{S}}_t = w + \int_0^t \widetilde{\boldsymbol{\theta}}_u d\widetilde{\mathbf{S}}_u - \int_0^t c_u du \geq 0 \tag{14}$$

$$\widetilde{\boldsymbol{\theta}}_T \cdot \widetilde{\mathbf{S}}_T = Z \tag{15}$$

- So, the problem of the consumer/investor is to choose the *controls* $\left(c, Z, \widetilde{\boldsymbol{\theta}}\right)$ to maximize $U(c, Z)$ subject to (14) and (15).

- That is:

$$\sup_{\left(c,Z,\widetilde{\boldsymbol{\theta}}\right)} U\left(c,Z\right) \tag{16}$$

- It is convenient to reformulate the problem in terms of "fraction of wealth invested in each security."
- This allow us to leave the "prices" out of the control.
- Let the wealth at time t be denoted by

$$W_t = \widetilde{\boldsymbol{\theta}}_t \cdot \widetilde{\mathbf{S}}_t$$

- This is going to be our *state variable.*
- So, define the control

$$\vartheta_t^i = \begin{cases} \theta_t^i S_t^i / W_t & \text{if} \quad W_t \neq 0 \\ 0 & \text{if} \quad W_t = 0 \end{cases} \tag{17}$$

- Let

$$\boldsymbol{\lambda} = \boldsymbol{\mu} - r\mathbf{1}_d$$

- Then, from the budget constraint the process for wealth is given by

$$dW_t = \left[W_t\left(\boldsymbol{\vartheta}_t \cdot \boldsymbol{\lambda} + r\right) - c_t\right]dt + W_t \boldsymbol{\vartheta}_t \boldsymbol{\sigma} d\mathbf{B}_t \tag{18}$$

- with initial condition $W_0 = w$.
- Remark: Since $W_t \geq 0$ for all t, we have that if $W_s = 0$ for some s, then $\boldsymbol{\vartheta}_t = c_t = 0$ for $t \geq s$.

7.1 The Bellman Equation

- Let the investor's *indirect utility function for wealth* at time t be

$$J(W_t, t) = E_t \left[\int_t^T u(c_u, u)\, du + F(Z_T) \right]$$

- Clearly, it is only a function of the state variable W_t (we have already optimized over c, Z and $\boldsymbol{\vartheta}$)
- The Hamilton - Jacoby - Bellman equation is given by

$$\sup_{(c,\boldsymbol{\vartheta})} u(c,t) + \mathcal{D}J(w,t) = 0 \qquad (19)$$

- where

$$\begin{aligned} \mathcal{D}J(w,t) &= J_t(w,t) + J_w(w,t)\left[w\boldsymbol{\vartheta}_t\boldsymbol{\lambda} + wr - c_t\right] \\ &\quad + \frac{1}{2} J_{ww}(w,t)\, w^2 \boldsymbol{\vartheta}_t \boldsymbol{\sigma}\boldsymbol{\sigma}' \boldsymbol{\vartheta}_t' \end{aligned}$$

- with boundary condition

$$J(w,T) = F(w)$$

- Why this form of the Bellman Equation?
- In discrete time over a time interval Δ we would have something like

$$J(w,t) = \max_{c \in R, \vartheta \in R^d} u(c,t)\,\Delta + E\left[J(W_{t+\Delta}, t+\Delta) \,|W_t = w\right]$$

- subject to a dynamic budget constraint.
- This implies that

$$\max_{c\in R,\vartheta\in R^d} u(c,t)\Delta + E_t\left[J\left(W_{t+\Delta}, t+\Delta\right) - J(w,t)\right] = 0$$

- Dividing by Δ and taking the limit as $\Delta \longrightarrow 0$, heuristically we have

$$\max_{c\in R,\vartheta\in R^d} u(c,t) + \frac{E_t\left[dJ_t\right]}{dt} = 0$$

- Clearly, by Ito's lemma $E_t\left[dJ_t\right]/dt = \mathcal{D}J$.
- Assuming strict concavity of the utility functions, we can now take the First Order Conditions of the maximization in (19).
- With respect to consumption

$$u_c(c_t,t) - J_w(w,t) = 0 \tag{20}$$

- With respect to $\boldsymbol{\vartheta}_t$

$$J_w(w,t)\boldsymbol{\lambda} + J_{ww}(w,t)\, w\boldsymbol{\sigma\sigma}'\boldsymbol{\vartheta}_t' = \mathbf{0} \tag{21}$$

- Let $\mathcal{I}_u(.,t)$ be the inverse of $u_c(.,t)$, that is, be such that for every x we have $\mathcal{I}_u\left(u_c(x,t),t\right) = x$.
- Then (20) implies

$$c_t^* = \mathcal{I}_u\left(J_w(w,t),t\right) \tag{22}$$

- Similarly, (21) implies

$$\boldsymbol{\vartheta}_t' = -\frac{J_w(w,t)}{wJ_{ww}(w,t)}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}(\boldsymbol{\mu} - r\mathbf{1}_d) \tag{23}$$

- Notice that the optimal portfolio weights are given by a vector $(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}(\boldsymbol{\mu} - r\mathbf{1}_d)$ which multiplies the Arrow-Pratt measure of relative risk tolerance (reciprocal of relative risk aversion) of the indirect utility function $J(w,t)$.

7.1.1 Example 1: Utility from final wealth

- Let $F(w) = \frac{w^{1-\gamma}}{1-\gamma}$ and $u(c_t,t) = 0$.
- Conjecture:

$$J(w,t) = k(t)\frac{w^{1-\gamma}}{1-\gamma}$$

- This form stems from the homogeneity of the utility function $F(w)$ and the linearity of the expectation operator and the stochastic integral.
- Define Z_T as the optimal level of consumption at time T when the initial wealth $= 1$.
- Consider now another wealth level w.
- It is justified to guess that $\widehat{Z}_T = wZ_T$, because

  1. $\widehat{Z}_T$ is budget feasible, because if $Z_T = \widetilde{\boldsymbol{\theta}}_T \cdot \widetilde{\mathbf{S}}_T$ can be obtained from 1, then $\widehat{Z}_T = wZ_T = w\widetilde{\boldsymbol{\theta}}_T \cdot \widetilde{\mathbf{S}}_T$ can be obtained from w due to the linearity of the stochastic integral.

2. In addition, suppose there exists an alternative consumption Z_T^* that is budget feasible with initial wealth w but with

$$E_t\left[\frac{(Z_T^*)^{1-\gamma}}{1-\gamma}\right] > E_t\left[\frac{(wZ_T)^{1-\gamma}}{1-\gamma}\right]$$

   – Divide through by $w^{1-\gamma}/(1-\gamma)$ to find

$$E_t\left[\frac{\left(\frac{Z_T^*}{w}\right)^{1-\gamma}}{1-\gamma}\right] > E_t\left[\frac{(Z_T)^{1-\gamma}}{1-\gamma}\right]$$

   – Clearly, Z_T^*/w can be financed by the initial 1 because (again) of the linearity of the stochastic integral.
   – But this contradicts that Z_T is optimal given $W_0 = 1$.

- Hence, we obtain

$$J(w,0) = E\left[\frac{(wZ)^{1-\gamma}}{1-\gamma}\right] = K\frac{w^{1-\gamma}}{1-\gamma}$$

- where $K = E\left[Z^{1-\gamma}\right]$.
- By repeating the same argument for all t, we more generally have

$$J(w,t) = k(t)\frac{w^{1-\gamma}}{1-\gamma}$$

- We can now solve for the optimal portfolio choice: We have

$$\begin{aligned} J_w &= k(t)\, w^{-\gamma} \\ J_{ww} &= -k(t)\,\gamma w^{-\gamma-1} \end{aligned}$$

- Substitute into the formula (23) to find

$$\begin{aligned}\boldsymbol{\vartheta}_t' &= -\frac{J_w(w,t)}{wJ_{ww}(w,t)}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\left(\boldsymbol{\mu}-r\mathbf{1}_d\right)\\ &= \frac{1}{\gamma}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\left(\boldsymbol{\mu}-r\mathbf{1}_d\right)\end{aligned}$$

- Hence, the *portfolio holdings are constant* and the percentage of wealth invested in the risky assets decrease with risk aversion γ.

- For $d = 1$ (only one risky asset) we have the standard formula

$$\vartheta_t = \frac{\mu - r}{\gamma\sigma^2}$$

- This rule give rise to two asset allocation puzzles:
  - First, the fraction invested in stocks is independent of age t, and thus of remaining life $T - t$. This is in contrast with both empirical evidence, that shows an inverted U shaped allocation to stocks with respect to age, and the typical recommendation of portfolio advisors.
  - Second, it predicts too high investment in stocks. Using unconditional averages, $\mu = 7\%$ and $\sigma = 16\%$, we obtain

Table: Portfolio Allocation

	Risk Aversion γ				
	2	4	6	8	10
ϑ	136%	68%	45%	34 %	27 %

- In contrast, depending on estimates, typical household holds betwee 6 % to 20 % in equity. Conditional on participating to the stock market, these number increase to about 40% of financial assets.

- Going back to the solution, note that $u = 0$ implies (of course) that $c_t^* = 0$.

- Finally, we obtain an explicit formula for $k(t)$. Plug back everything into the Bellman equation and impose equality to zero.

- That is, impose

$$\begin{aligned} 0 &= u(c,t) + J_t(w,t) + J_w(w,t)\left[w\boldsymbol{\vartheta}_t\boldsymbol{\lambda}+wr-c_t\right] \\ &+\frac{1}{2}J_{ww}(w,t)\,w^2\boldsymbol{\vartheta}_t\boldsymbol{\sigma}\boldsymbol{\sigma}'\boldsymbol{\vartheta}_t' \end{aligned}$$

- and we obtain

$$k'(t)\frac{1}{1-\gamma}+k(t)\left(\frac{1}{2\gamma}\boldsymbol{\lambda}'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\lambda}+r\right)=0$$

- or

$$k'(t) = -qk(t)$$

- where

$$q = (1-\gamma)\left(\frac{1}{2\gamma}\boldsymbol{\lambda}'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\lambda}+r\right)$$

- The boundary condition is of course $k(T) = 1$. Hence

$$k(t) = e^{q(T-t)}$$

- which yields

$$J(w,t) = e^{q(T-t)}\frac{w^{1-\gamma}}{1-\gamma}$$

- **We are not done yet!**
- We need to *verify* that this solution is optimal. That is, we need to verify that

  1. For every admissible control $(c, \boldsymbol{\vartheta})$

  $$E\left[\int_0^T u(c_t,t)\,dt + F(W_T)\right] \leq J(w,0)$$

  2. Equality is obtained by using the solution provided.

- To show 1, let $(c, \boldsymbol{\vartheta})$ be an arbitrary admissible control for initial wealth w and let $\{W_t\}$ be the process solving the budget constraint

$$dW_t = \left[W_t\left(\boldsymbol{\vartheta}_t \cdot \boldsymbol{\lambda}+r\right) - c_t\right]dt + W_t\boldsymbol{\vartheta}_t\boldsymbol{\sigma}d\mathbf{B}_t \qquad (24)$$

- By the Bellman equation (19) we have

$$u(c,t) + \mathcal{D}J(w,t) \leq 0 \tag{25}$$

- (it was 0 at its maximum)
- By Ito's Lemma

$$\begin{aligned} J(W_T,T) &= J(w,0) + \int_0^T J_t(W_t,t)\,dt + \int_0^T J_w(W_t,t)\,dW_t \\ &\quad + \frac{1}{2}\int_0^T J_{ww}(W_t,t)\,(dW_t)^2 \\ &= J(w,0) + \int_0^T \mathcal{D}J(W_t,t)\,dt \\ &\quad + \int_0^T J_w(W_t,t)\,W_t\boldsymbol{\vartheta}_t\boldsymbol{\sigma}\,d\mathbf{B}_t \\ &\leq J(w,0) + \int_0^T \boldsymbol{\beta}_t d\mathbf{B}_t \end{aligned}$$

- where $\boldsymbol{\beta}_t = J_w(W_t,t)\,W_t\boldsymbol{\vartheta}_t\boldsymbol{\sigma}$, and the inequality is due to $\int_0^T \mathcal{D}J(W_t,t)\,dt \leq 0$.
- Hence, since we have the boundary condition $J(W_T,T) = F(W_T)$ we finally obtain

$$F(W_T) \leq J(w,0) + \int_0^T \boldsymbol{\beta}_t d\mathbf{B}_t$$

- Taking expectations on both sides, we find

$$E_0\left[F\left(W_T\right)\right] \leq J\left(w,0\right) + E_0\left[\int_0^T \boldsymbol{\beta}_t d\mathbf{B}_t\right] \leq J\left(w,0\right)$$

- where the last inequality stems from $\int_0^T \boldsymbol{\beta}_t d\mathbf{B}_t$ being a non-negative local martingale and hence a super-martingale (see TN#0).

- To show 2, just do again the calculations with $u\left(c,t\right) + \mathcal{D}J\left(w,t\right) = 0$ this time.

- In addition, given the explicit formula for $J\left(w,t\right)$ one can show that

$$\boldsymbol{\beta}_t = J_w\left(W_t,t\right) W_t \boldsymbol{\vartheta}_t \boldsymbol{\sigma} = k\left(t\right) w^{1-\gamma} \boldsymbol{\vartheta}_t \boldsymbol{\sigma} \in \mathcal{H}^1$$

- Hence, now $E_0\left[\int_0^T \boldsymbol{\beta}_t d\mathbf{B}_t\right] = 0$ because $\int_0^T \boldsymbol{\beta}_t d\mathbf{B}_t$ is in fact a martingale, leaving us with

$$E\left[F\left(W_T\right)\right] = J\left(w,0\right)$$

7.1.2 Example 2: Utility from Consumption Stream with Infinite Horizon

- A second standard example is the case where

$$U\left(c\right) = E\left[\int_0^\infty e^{-\phi t} u\left(c_t\right) dt\right] \tag{26}$$

and

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

- The Bellman equation is now

$$\sup_{(c,\boldsymbol{\vartheta})} u(c) - \phi J(w) + \mathcal{D}J(w) = 0 \tag{27}$$

- where

$$\mathcal{D}J(w,t) = J_w(w)\left[w\boldsymbol{\vartheta}_t\boldsymbol{\lambda} + wr - c_t\right] + \frac{1}{2}J_{ww}(w)\,w^2\boldsymbol{\vartheta}_t\boldsymbol{\sigma}\boldsymbol{\sigma}'\boldsymbol{\vartheta}_t'$$

- In addition, we do not have a boundary condition anymore.
- Instead, a transversality condition is imposed

$$\lim_{T\longrightarrow\infty} E\left(e^{-\phi T}|J(W_T)|\right) = 0$$

- Again, the FOC for consumption and portfolio choice are

$$\begin{aligned} u_c(c) &= J_w(w) \\ \boldsymbol{\vartheta}_t' &= -\frac{J_w(w)}{wJ_{ww}(w)}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\lambda} \end{aligned}$$

- Similar arguments as before lead us to conjecture

$$J(w) = K\frac{w^{1-\gamma}}{1-\gamma}$$

- Hence,

$$c_t^{-\gamma} = KW_t^{-\gamma} \implies c_t = K^{-\frac{1}{\gamma}} W_t$$

- As before, the optimal portfolio rule is

$$\boldsymbol{\vartheta}_t' = \frac{1}{\gamma} \left(\boldsymbol{\sigma}\boldsymbol{\sigma}' \right)^{-1} \boldsymbol{\lambda}$$

- Plug everything back into the Bellman equation

$$0 = u\left(c\right) - \phi J\left(w\right) + J_w\left(w\right)\left[w\boldsymbol{\vartheta}_t\boldsymbol{\lambda} + wr - c_t\right] + \frac{1}{2}J_{ww}\left(w\right)w^2\boldsymbol{\vartheta}_t\boldsymbol{\sigma}\boldsymbol{\sigma}'\boldsymbol{\vartheta}_t'$$

- to find

$$\begin{aligned} 0 \;=\; & K^{-\frac{1-\gamma}{\gamma}} \frac{w^{1-\gamma}}{1-\gamma} - \phi K \frac{w^{1-\gamma}}{1-\gamma} \\ & + K w^{-\gamma} \left[w \frac{1}{\gamma} \boldsymbol{\lambda}' \left(\sigma\sigma' \right)^{-1} \boldsymbol{\lambda} + wr - K^{-\frac{1}{\gamma}} w \right] \\ & - \frac{1}{2} \gamma w^{-\gamma-1} w^2 \frac{1}{\gamma^2} \boldsymbol{\lambda}' \left(\sigma\sigma' \right)^{-1} \boldsymbol{\lambda} \end{aligned}$$

- Deleting terms, we obtain that the constant K must be

$$K = \left(\frac{\phi - (1-\gamma) r}{\gamma} - \frac{1-\gamma}{2\gamma^2} \boldsymbol{\lambda}' \left(\sigma\sigma' \right)^{-1} \boldsymbol{\lambda} \right)^{-\gamma}$$

- The only thing left to show is that the transversality condition holds.
- This can be easily done by using the definition of $J\left(W_T\right)$ and Ito's Lemma.

8 The Martingale Approach

- In the next few classes we shall use the martingale approach to solve a number of interesting cases.
- It is therefore important to understand the method in the simplest case (Merton's problem).
- The idea is to transform the dynamic problem in a *static problem*, where we *choose a function* within a set of feasible ones to maximize a *functional* subject to some (static) constraint.
- Let D be the set of consumption pairs (c, Z) where $c \geq 0$ is adapted and $\int_0^T c_t dt < \infty$ a.s. and $Z \geq 0$ is $\mathcal{F}_T$ measurable.
- The objective function is

$$\sup_{(c,Z,\boldsymbol{\theta})\in\Gamma(w)} U(c, Z) \tag{28}$$

- where

$$\begin{aligned} U(c,Z) &= E\left[\int_0^T u(c_t,t) + F(Z)\right] \\ \Gamma(w) &= \left\{ \begin{array}{c} (c,Z,\boldsymbol{\theta}) : (c,Z) \in \widehat{D} \text{ and} \\ (c,Z,\boldsymbol{\theta}) \text{ is budget feasible} \end{array} \right\} \\ \widehat{D} &= \left\{ \begin{array}{c} (c,Z) \in D : E\left(\int_0^T c_t^2 dt\right) < \infty \\ \text{and } E\left(Z^2\right) < \infty \end{array} \right\} \end{aligned}$$

- Consider first the deflated prices and consumption

$$\widehat{\mathbf{S}}_t = e^{-rt}\mathbf{S}_t;\ \widehat{c}_t = e^{-rt}c_t \text{ and } \widehat{Z} = e^{-rt}Z$$

- From the Numeraire Invariance Theorem we obtain immediately that a strategy $\left(c, Z, \left(\theta^0, \boldsymbol{\theta}\right)\right)$ is budget feasible with respect to $\mathbf{S}$ if and only if $\left(\widehat{c}, \widehat{Z}, \left(\theta^0, \boldsymbol{\theta}\right)\right)$ is budget feasible with respect to $\widehat{\mathbf{S}}$, which implies

$$\theta_t^0 \cdot 1 + \boldsymbol{\theta}_t \cdot \widehat{\mathbf{S}}_t = w + \int_0^t \boldsymbol{\theta}_u \cdot d\widehat{\mathbf{S}}_u - \int_0^t \widehat{c}_u du \geq 0 \quad (29)$$

$$\theta_T^0 + \boldsymbol{\theta}_T \cdot \widehat{\mathbf{S}}_T = \widehat{Z} \quad (30)$$

- Define now the constant

$$\boldsymbol{\nu} = \boldsymbol{\sigma}^{-1} \left(\boldsymbol{\mu} - r \mathbf{1}_d \right)$$

- and let

$$\xi_t = \exp\left(-\boldsymbol{\nu}' \mathbf{B}_t - \frac{t}{2} \boldsymbol{\nu}' \boldsymbol{\nu} \right)$$

- Since the Novikov's condition is trivially satisfied, since $\boldsymbol{\nu}$ is a constant, there is an equivalent martingale measure Q such that $\widehat{S}$ is a $Q-$martingale.
- We then have the first key result, namely, the transformation of the dynamic budget constraint into a static one (under Q).

- **Proposition 4**: Let $w > 0$ be given. Then $(c, Z, \boldsymbol{\theta})$ is budget feasible if and only if

$$E^Q\left(e^{-rt}Z + \int_0^T e^{-rt}c_t dt\right) \leq w \tag{31}$$

- This result, which can be greatly generalized, is the starting point of the Martingale Solution.
- The proof is in the appendix.

8.1 The Martingale Solution

- The proposition in the previous section leads us to the following maximization problem

$$\sup_{(c,Z)\in\widehat{D}} U(c, Z) \tag{32}$$

- subject to

$$E^Q\left(e^{-rT}Z + \int_0^T e^{-rt}c_t dt\right) \leq w$$

- We use the following known result to solve this maximization problem
- **Saddle Point Theorem**: Let $U : X \longrightarrow \mathcal{R}$ be concave on a convex set X and let $g : X \longrightarrow \mathcal{R}^m$ be a convex function. Consider the program

$$\sup_{x\in X} U(x) \text{ subject to } g(x) \leq 0 \tag{33}$$

- Define the Lagrangian function

$$L(x,\lambda) = U(x) - \lambda \cdot g(x)$$

1. Suppose there exists $\underline{x} \in X$ such that $g(\underline{x}) << 0$ (*Slater condition*). Then if x_0 solves (33), then there exists $\lambda_0 \in \mathcal{R}^m$ such that for all $(x,\lambda) \in \mathcal{R} \times \mathcal{R}^m$

$$L(x,\lambda_0) \leq L(x_0,\lambda_0) \leq L(x_0,\lambda)$$

   [(x_0,λ_0) is a *saddle point*]

2. If (x_0,λ_0) is a *saddle point*, then x_0 solves (33)

- We apply this result to our problem.

- Specifically, we solve

$$\sup_{(c,Z)\in\widehat{D}} U(c,Z) - \lambda E^Q\left(e^{-rT}Z + \int_0^T e^{-rt}c_t dt - w\right) \quad (34)$$

- with the complementary slackness condition

$$E^Q\left(e^{-rT}Z^* + \int_0^T e^{-rt}c_t^* dt\right) = w$$

- It is convenient to rewrite all the expectations involved under the same probability measure.

- Let's choose P as the reference probability measure.

- We know that if ξ_T is the Radon-Nikodym derivative

$$\xi_T = \frac{dQ}{dP}$$

- for any random variable Z with $E^Q[|Z|] < \infty$ we have

$$E^Q[Z] = E[\xi_T Z]$$

- Define the *state-price deflator*

$$\pi_t = e^{-rt}\xi_t$$

- We then have the following proposition:

- **Proposition 5**: There is a trading strategy $\theta^* \in \mathcal{L}(\mathbf{S})$ such that (c^*, Z^*, θ^*) solves Merton's problem (28) (or (32)) if and only if

  1. $$E\left(\int_0^T \pi_t c_t^* dt + \pi_T Z^*\right) = w \tag{35}$$

  2. there is a constant $\lambda^* > 0$ such that (c^*, Z^*) solves

  $$\sup_{(c,Z)\in\widehat{D}} L(c, Z; \lambda^*) \tag{36}$$

  where

  $$L(c, Z; \lambda^*) = E\left(\int_0^T [u(c_t, t) - \lambda\pi_t c_t]\, dt + F(Z) - \lambda\pi_T Z\right) \tag{37}$$

- This is direct consequence of the Saddle Point Theorem once one shows that indeed (35) is equivalent to the constraint in Merton's problem (32), that is

$$E^{Q}\left(\int_0^T e^{-rt}c_t dt + e^{-rT}Z\right) = w \tag{38}$$

- To show it, we need the following result:

- **Result:** *Fubini Theorem*: Let X_t be a measurable process defined on a probability space $(\Omega, \mathcal{F}, P)$. If

$$E\left(\int_0^T |X_t| dt\right) < \infty$$

Then

$$E\left(\int_0^T X_t dt\right) = \int_0^T E\left[X_t\right] dt$$

- That is, we can change the order of integration.

- Then, we have the following equalities

$$\begin{aligned}
E^{Q}\left(\int_0^T e^{-rt}c_t dt + e^{-rT}Z\right) &= E\left(\xi_T\left(\int_0^T e^{-rt}c_t dt + e^{-rT}Z\right)\right) \\
&= E\left(\int_0^T \xi_T e^{-rt}c_t dt + e^{-rT}\xi_T Z\right) \\
\text{(Law It. Exp. + Fubini)} &= E\left(\int_0^T E_t\left(\xi_T e^{-rt}c_t\right) dt + e^{-rT}\xi_T Z\right) \\
(c_t \text{ known at } t) &= E\left(\int_0^T E_t\left(\xi_T\right) e^{-rt}c_t dt + e^{-rT}\xi_T Z\right)
\end{aligned}$$

$$(\xi_t \text{ is a martingale}) = E\left(\int_0^T e^{-rt}\xi_t c_t dt + e^{-rT}\xi_T Z\right)$$
$$= E\left(\int_0^T \pi_t c_t dt + \pi_T Z\right)$$

- This proves the proposition.

8.1.1 The Solution to the Maximization Problem

- We are left with solving the (static) optimization problem (36).
- It turns out that we can simply use standard First Order Conditions as if all the integrals involved are just "finite sums."
- That is, take the FOC of the Lagrangian function (37) with respect to c and Z

$$u_c\left(c_t^*, t\right) - \lambda\pi_t = 0$$
$$F'\left(Z\right) - \lambda\pi_T = 0$$

- We can solve for consumption first, which yields

$$c_t^* = \mathcal{I}_u\left(\lambda\pi_t, t\right) \tag{39}$$
$$Z^* = \mathcal{I}_Z\left(\lambda\pi_T\right) \tag{40}$$

- where $\mathcal{I}_u\left(., t\right)$ and $\mathcal{I}_Z\left(., t\right)$ denote the inverse of the marginal utility functions $u_c\left(., t\right)$ and $F\left(.\right)$.

- We have still to determine the necessary condition for λ^*. As usual, we make use of the budget constraint (35) at this time.

- Define the function

$$\widehat{w}(\lambda) = E\left(\int_0^T \pi_t \mathcal{I}_u(\lambda \pi_t, t)\, dt + \pi_T \mathcal{I}_Z(\lambda \pi_T)\right) \tag{41}$$

- The budget constraint requires

$$\widehat{w}(\lambda) = w \tag{42}$$

- This equation defines the optimal Lagrangian multiplier λ^*.

- Notice that so far we have only solved for the optimal consumption (c^*, Z^*) for Merton problem.

- What happened to the optimal investment strategy?
- This is an implication of the assumption of complete markets.
- Since we know that for any given consumption plan (c, Z) there exists a trading strategy that finances it, we have first solved directly for the optimal consumption plans without caring about the optimal trading strategy.
- The latter can be found as a residual once we know (c^*, Z^*) (see below for an example).

8.1.2 Existence and the Inada Conditions

- To ensure the existence of a solution to Merton problem, we must impose some structure to the functions $u(., t)$ and $F(.)$.
- A strictly concave, increasing function $F : \mathcal{R}_+ \longrightarrow \mathcal{R}$ that is differentiable on $(0, \infty)$ satisfies the *Inada Conditions* if

$$\begin{aligned} \lim_{x \downarrow 0} F'(x) &= \infty \\ \lim_{x \uparrow \infty} F'(x) &= 0 \end{aligned}$$

- If F satisfies the Inada Conditions, then the inverse I_F of $F'(.)$ is well defined as a strictly decreasing continuous function from $(0, \infty)$ to $(0, \infty)$.
- The following condition is often imposed.

- **Condition A**:

  1. Either $F = 0$ or F is (i) differentiable on $(0, \infty)$, strictly concave and satisfies the Inada conditions.
  2. Either $u = 0$ or for every t, $u(., t)$ is differentiable on $(0, \infty)$, strictly concave and satisfies the Inada conditions.
  3. Either u or F is non zero.
  4. For each λ, $\widehat{w}(\lambda) < \infty$

- **Theorem**: Under condition A, for every initial wealth $w > 0$, Merton problem (28) (or (32)) has a solution (c^*, Z^*, θ^*), where (c^*, Z^*) is given by (39) and (40) for a unique $\lambda \in (0, \infty)$

8.2 Example 1: Utility from Final Wealth

- To see the martingale approach in action, let's consider again the case where $u = 0$ and $F(Z) = \frac{Z^{1-\gamma}}{1-\gamma}$

- Clearly, we still have $c_t^* = 0$. In addition, from the first order conditions we have

$$Z^* = (\lambda \pi_T)^{-\frac{1}{\gamma}} \tag{43}$$

- where, we recall, we have

$$\pi_t = e^{-rt}\xi_t = \exp\left(-\left(r + \frac{1}{2}\boldsymbol{\nu}'\boldsymbol{\nu}\right)t - \boldsymbol{\nu}'\mathbf{B}_t\right) \tag{44}$$

- and

$$\boldsymbol{\nu} = \boldsymbol{\sigma}^{-1} \left(\boldsymbol{\mu} - r\mathbf{1}_d \right)$$

- We can substitute this into the equation for $\widehat{w}(\lambda)$ (41)

$$\widehat{w}(\lambda) = E\left(\pi_T \left(\lambda \pi_T \right)^{-\frac{1}{\gamma}} \right) \tag{45}$$

- We can impose the budget constraint condition $\widehat{w}(\lambda) = w$

$$w = E\left(\lambda^{-\frac{1}{\gamma}} \pi_T^{1-\frac{1}{\gamma}} \right) = \lambda^{-\frac{1}{\gamma}} E\left(\pi_T^{\frac{\gamma-1}{\gamma}} \right) \tag{46}$$

- which yields

$$\lambda^* = \left\{ \frac{1}{w} E\left(\pi_T^{\frac{\gamma-1}{\gamma}} \right) \right\}^{\gamma}$$

- Notice that from (44) we have that for every constant β:

$$\begin{aligned} E\left(\pi_T^{\beta} \right) &= E\left(\exp\left(-\beta \left(r + \frac{1}{2} \boldsymbol{\nu}' \boldsymbol{\nu} \right) T - \beta \boldsymbol{\nu}' \mathbf{B}_t \right) \right) \\ &= \exp\left(-\beta r T - \beta \frac{1}{2} \boldsymbol{\nu}' \boldsymbol{\nu} T + \beta^2 \frac{1}{2} \boldsymbol{\nu}' \boldsymbol{\nu} T \right) \\ &= \exp\left(-\beta r T - \beta \left(1 - \beta \right) \frac{1}{2} \boldsymbol{\nu}' \boldsymbol{\nu} T \right) \end{aligned}$$

- Hence, setting $\beta = \frac{\gamma-1}{\gamma}$ we have

$$E\left(\pi_T^{\frac{\gamma-1}{\gamma}} \right) = \exp\left(-\frac{\gamma - 1}{\gamma} \left(r + \frac{1}{2\gamma} \boldsymbol{\nu}' \boldsymbol{\nu} \right) T \right)$$

- which implies

$$\lambda^* = \left\{ \frac{1}{w} \exp\left(\frac{(1-\gamma)}{\gamma} \left(r + \frac{1}{2\gamma} \boldsymbol{\nu}'\boldsymbol{\nu} \right) T \right) \right\}^{\gamma}$$

- Substitute back into (43) to obtain

$$\begin{aligned} Z^* &= w \exp\left(\frac{(\gamma-1)}{\gamma} \left(r + \frac{1}{2\gamma} \boldsymbol{\nu}'\boldsymbol{\nu} \right) T \right) \pi_T^{-\frac{1}{\gamma}} && (47) \\ &= w \exp\left(\left(1 - \frac{1}{\gamma}\right) \left(r + \frac{1}{2\gamma} \boldsymbol{\nu}'\boldsymbol{\nu} \right) T \right) && (48) \\ &\quad \times \exp\left(\frac{1}{\gamma} \left(r + \frac{1}{2} \boldsymbol{\nu}'\boldsymbol{\nu} \right) T + \frac{1}{\gamma} \boldsymbol{\nu}' \mathbf{B}_T \right) && (49) \\ &= w \exp\left(rT + \frac{1}{\gamma} \boldsymbol{\nu}'\boldsymbol{\nu} T - \frac{1}{2\gamma^2} \boldsymbol{\nu}'\boldsymbol{\nu} T + \frac{1}{\gamma} \boldsymbol{\nu}' \mathbf{B}_T \right) && (50) \end{aligned}$$

- Noitce that this Brownian motion is under P. Due to Girsanov's theorem, we can define a Brownian motion under Q as

$$\widehat{\mathbf{B}}_t = \mathbf{B}_t + \boldsymbol{\nu} t$$

- Hence, we can rewrite

$$Z^* = w e^{rT} \exp\left(-\frac{1}{2\gamma^2} \boldsymbol{\nu}'\boldsymbol{\nu} T + \frac{1}{\gamma} \boldsymbol{\nu}' \widehat{\mathbf{B}}_T \right)$$

- From Novikov's theorem, the exponent part of this random variable is a martingale under Q.

- The last two questions are: (1) How does Z^* relates to the optimal wealth and (2) what is the trading strategy that generates the optimal consumption level.

- We answer the two questions simultaneously, by making use (constructively) of the propositions above.

- Let $\left(\theta^0, \boldsymbol{\theta}\right)$ be a trading strategy that finances Z. Since we know this is budget feasible with respect to $(\beta, \mathbf{S})$, it is also budget feasible with respect to $\left(1, \widehat{\mathbf{S}}\right)$ with $\widehat{\mathbf{S}} = \mathbf{S}/\beta$.

- Hence, we can define the wealth at time t as

$$\widehat{W}_t = \theta_t^0 + \theta_t \cdot \widehat{\mathbf{S}}_t = E_t^Q \left[e^{-rT} Z^*\right] \tag{51}$$

- with $W_0 = \theta_0^0 + \theta_0 \cdot \widehat{\mathbf{S}}_0 = w$ (from (38) equation (51) holds at time 0. A similar proof, as in the one of Proposition 4, shows it for all t).

- First, we know that

$$\begin{aligned} \widehat{W}_t &= \theta_t^0 + \theta_t \cdot \widehat{\mathbf{S}}_t = w + \int_0^t \boldsymbol{\theta}_u d\widehat{\mathbf{S}}_u & (52) \\ &= w + \int_0^t \boldsymbol{\theta}_u \cdot \mathbf{I}_{\widehat{S}} \boldsymbol{\sigma} d\widehat{\mathbf{B}}_u & (53) \end{aligned}$$

- where we use the fact that we can rewrite

$$d\widehat{\mathbf{S}}_t = \mathbf{I}_{\widehat{S}} \boldsymbol{\sigma} d\widehat{\mathbf{B}}_t$$

- (Recall that $\mathbf{I}_{\widehat{S}}$ is the diagonal matrix with $\widehat{S}_t^i$ on the $ii-$th element).

- Second, we also know (see proof of Proposition 4) that under Q the scaled wealth $\widehat{W} = W/\beta$ is a martingale, which implies

$$\begin{aligned}
\widehat{W}_t &= E_t^Q \left[e^{-rT} Z^* \right] \\
&= E_t^Q \left[e^{-rT} w e^{rT} \exp\left(-\frac{1}{2\gamma^2} \boldsymbol{\nu}' \boldsymbol{\nu} T + \frac{1}{\gamma} \boldsymbol{\nu}' \widehat{\mathbf{B}}_T \right) \right] \\
&= w E_t^Q \left[\exp\left(-\frac{1}{2\gamma^2} \boldsymbol{\nu}' \boldsymbol{\nu} T + \frac{1}{\gamma} \boldsymbol{\nu}' \widehat{\mathbf{B}}_T \right) \right] \\
&= w \exp\left(-\frac{1}{2\gamma^2} \boldsymbol{\nu}' \boldsymbol{\nu} t + \frac{1}{\gamma} \boldsymbol{\nu}' \widehat{\mathbf{B}}_t \right)
\end{aligned}$$

- By applying Ito's lemma to both sides, we have

$$\widehat{W}_t = w + \int_0^t \widehat{W}_u \frac{1}{\gamma} \boldsymbol{\nu}' d\widehat{\mathbf{B}}_u \tag{54}$$

- Comparing (54) with (53) and recalling that $\boldsymbol{\nu} = \boldsymbol{\sigma}^{-1} (\boldsymbol{\mu} - r\mathbf{1}_d)$ we see that

$$\widehat{W}_u \frac{1}{\gamma} \left(\boldsymbol{\sigma}^{-1} (\boldsymbol{\mu} - r\mathbf{1}_d) \right)' = \boldsymbol{\theta}_u I_{\widehat{S}} \boldsymbol{\sigma}$$

- This yields the portfolio weights

$$\boldsymbol{\vartheta}' = \frac{I_S \boldsymbol{\theta}_u'}{W_u} = \frac{1}{\gamma} (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1} (\boldsymbol{\mu} - r\mathbf{1}_d)$$

- These should be confronted with (17).

8.3 Generalization

- The approach so far looks rather messy.
- However, the power of the result is that the existence theorem and the characterization hold under very general conditions.
- The Bellman equation approach has the problem that one needs to "guess" the form of the value function to obtain results. Sometimes, it is very hard to guess it if $u(C_t, t)$ is not power utility.
- Indeed, if we want to study the consumption implications of habit formation, for instance, under complete markets, we can readily apply the martingale approach, but it is much harder to guess the Bellman equation in this case. We will see applications of this approach later on.
- In addition, when we insert market completeness and portfolio constraints, it turns out that we can use a similar procedure.
- The above result holds under the following much more general conditions about the security prices.
- Let $(\beta, \mathbf{S})$ be processes of the form

$$d\beta_t = r_t \beta_t dt \text{ with } \beta_0 > 0 \tag{55}$$

$$dS_t^i = \mu_t^i S_t^i dt + S_t^i \boldsymbol{\sigma}_t^i d\mathbf{B}_t \text{ with } S_0^i > 0 \tag{56}$$

- where r, $\mu = \left(\mu^1, ..., \mu^d\right)$ and $\boldsymbol{\sigma}^i$ are bounded, adapted processes.
- Assume again that $\{\boldsymbol{\sigma}_t\}$ is invertible almost everywhere.

- Define by $\boldsymbol{\lambda}_t = \boldsymbol{\mu}_t - r_t \mathbf{1}_d$ the excess return process.
- Let the $d \times 1$ market price of risk process

$$\boldsymbol{\nu}_t = \boldsymbol{\sigma}_t^{-1} \boldsymbol{\lambda}_t$$

- satisfy the Novikov's condition

$$E\left[\exp\left(\frac{1}{2}\int_0^T \boldsymbol{\nu}_t' \boldsymbol{\nu}_t dt\right)\right] < \infty$$

- and finally assume that $var\left(\xi_T\right) < \infty$, where

$$\xi_t = \exp\left(-\frac{1}{2}\int_0^t \boldsymbol{\nu}_u' \boldsymbol{\nu}_u du - \int_0^t \boldsymbol{\nu}_u' d\mathbf{B}_u\right)$$

- Notice that these assumptions imply
  1. Complete markets $(d = N)$ and σ is invertible;
  2. No arbitrage (there exists an equivalent martingale measure Q defined by ξ_T)
- Define the *state-price deflator*

$$\begin{aligned} \pi_t &= \exp\left(-\int_0^t r_u du\right)\xi_t \qquad (57) \\ &= \exp\left(-\int_0^t \left(r_u + \frac{1}{2}\boldsymbol{\nu}_u' \boldsymbol{\nu}_u\right) du - \int_0^t \boldsymbol{\nu}_u' d\mathbf{B}_u\right) \end{aligned}$$

- We then have that the optimal consumption/portfolio choice can still be reformulated as in proposition 5, where one uses (57) to formulate the static budget constraint.

- Again, we have that the solution for the optimal consumption is given by

$$c_t^* = \mathcal{I}_u(\lambda\pi_t, t) \tag{58}$$

$$Z^* = \mathcal{I}_F(\lambda\pi_T) \tag{59}$$

- where $\mathcal{I}_u$ and $\mathcal{I}_F$ are the inverse of the utility functions.

- In addition, by defining again

$$\widehat{w}(\lambda) = E\left(\int_0^T \pi_t \mathcal{I}_u(\lambda\pi_t, t)\,dt + \pi_T \mathcal{I}_Z(\lambda\pi_T)\right) \tag{60}$$

- the solution to λ^* is given by the equality $\widehat{w}(\lambda) = w$.

- Once again, one also obtains

- **Proposition 6**: Suppose $(\beta, \mathbf{S})$ are processes defined by (55) and (56). Under condition A, for any $w > 0$ there exists an optimal consumption policy given by (58) and (59) for a unique λ.

- The trouble is that we can generally only characterize consumption (up to a constant λ), but not the trading strategy.

- There are a few cases where explicit solutions can be found, but they are rare.

9 Appendix

- **Result:** If $\int_0^t \theta_s dB_s$ is a martingale, then

$$var\left(\int_0^T \theta_t dB_t\right) = E\left(\int_0^T \theta_t^2 dt\right)$$

  - *Proof of Proposition 3*: "IF". First, we immediately have that $N \geq d$ (otherwise $rank\left(\boldsymbol{\sigma}\right) < d$).
  - Let Y be any random variable with finite variance. Let $\mathbf{Z} = \mathbf{S}/\beta$ and let Q be an equivalent martingale measure with respect to $\mathbf{Z}$.
  - From Ito's lemma

$$d\mathbf{Z}_t = \left(-r_t\mathbf{Z}_t + \frac{\boldsymbol{\mu}_t}{\beta_t}\right)dt + \frac{\boldsymbol{\sigma}_t}{\beta_t}d\mathbf{B}_t$$

  - Since Q is an equivalent martingale measure for Z, there exists $\widehat{\mathbf{B}}$ under Q such that

$$d\mathbf{Z}_t = \frac{\boldsymbol{\sigma}_t}{\beta_t}d\widehat{\mathbf{B}}_t \tag{61}$$

  - Since by definition $\mathbf{S}_t = \mathbf{Z}_t \times \beta_t$, Ito's lemma shows

$$d\mathbf{S}_t = r_t dt + \boldsymbol{\sigma}_t d\widehat{\mathbf{B}}_t$$

  - (This by itself is an interesting result!)
  - Consider now the process

$$M_t = E^Q\left[\frac{Y}{\beta_T}|\mathcal{F}_t\right]$$

– This is a martingale under Q, and hence we can apply the following result:

– **Result:** *Martingale representation theorem:* For any (local) $Q-$martingale M_t, there exists a $d-$dimensional process $\boldsymbol{\eta} \in \mathcal{L}\left(\widehat{\mathbf{B}}\right)$ such that

$$M_t = M_0 + \int_0^t \boldsymbol{\eta}_u d\widehat{\mathbf{B}}_u$$

– Since $rank\left(\sigma\right) = d$, we also have that there exists a $N-$dimensional adapted process $\boldsymbol{\theta}$ that satisfies the system

$$\boldsymbol{\theta}_t \cdot \boldsymbol{\sigma}_t = \beta_t \boldsymbol{\eta}_t \tag{62}$$

– Finally, let

$$\theta_t^0 = M_0 + \int_0^t \boldsymbol{\theta}_s d\mathbf{Z}_s - \boldsymbol{\theta}_t \mathbf{Z}_t$$

– Notice that $\left(\theta^0, \boldsymbol{\theta}\right)$ is self-financing with respect to the (deflated) price process $(1, \mathbf{Z})$. In fact

$$\begin{aligned}
\left(\theta_t^0, \boldsymbol{\theta}_t\right) \cdot (1, \mathbf{Z}_t) &= M_0 + \int_0^t \boldsymbol{\theta}_s d\mathbf{Z}_s - \boldsymbol{\theta}_t \mathbf{Z}_t + \boldsymbol{\theta}_t \mathbf{Z}_t \\
&= \left(\theta_0^0, \boldsymbol{\theta}_0\right) \cdot (1, \mathbf{Z}_0) + \int_0^t \boldsymbol{\theta}_s d\mathbf{Z}_s \\
&= \left(\theta_0^0, \boldsymbol{\theta}_0\right) \cdot (1, \mathbf{Z}_0) + \int_0^t \left(\theta_s^0, \boldsymbol{\theta}_s\right)(0, d\mathbf{Z}_s)
\end{aligned}$$

– Also, by using (61)

$$\begin{aligned}\left(\theta_T^0, \boldsymbol{\theta}_T\right) \cdot (1, \mathbf{Z}_T) &= M_0 + \int_0^T \boldsymbol{\theta}_s d\mathbf{Z}_s = M_0 + \int_0^T \boldsymbol{\theta}_s \frac{\boldsymbol{\sigma}_s}{\beta_s} d\widehat{\mathbf{B}}_s \\ &= M_0 + \int_0^T \boldsymbol{\eta}_s d\widehat{\mathbf{B}}_s = M_T \\ &= E\left[\frac{Y}{\beta_T} | \mathcal{F}_T\right] = \frac{Y}{\beta_T}\end{aligned}$$

– Hence, markets are complete under the deflated price system $(1, \mathbf{Z})$. The numeraire invariance theorem implies that markets are complete under the price system $(\beta, \mathbf{S})$.

– "ONLY IF." Suppose now that it is not true that $rank\,(\sigma) = d$ (almost everywhere). Then (62) does not have any solution for some $\boldsymbol{\eta}_t \in \mathcal{L}\left(\widehat{\mathbf{B}}\right)$.

– It is possible to show that this implies that there exists no self-financing trading strategy $\left(\theta_t^0, \boldsymbol{\theta}_t\right)$ for $(1, \mathbf{Z})$ such that $\left(\theta_T^0, \boldsymbol{\theta}_T\right) \cdot (1, \mathbf{Z}_T) = \int_0^T \boldsymbol{\eta}_s d\widehat{\mathbf{B}}_s$ and hence no trading strategy with respect to $(\beta, \mathbf{S})$ yielding $\left(\theta_T^0, \boldsymbol{\theta}_T\right) \cdot (\beta_T, \mathbf{S}_T) = \beta_T \int_0^T \boldsymbol{\eta}_s d\widehat{\mathbf{B}}_s$.

– Define $Y = \beta_T \int_0^T \boldsymbol{\eta}_s d\widehat{\mathbf{B}}_s$ and the theorem is shown.□

- *Proof of Proposition 4*: ONLY IF. Suppose $(c, Z, \boldsymbol{\theta})$ is budget feasible. Then, there exists $\theta \in \mathcal{L}\left(\widehat{\mathbf{S}}\right)$ satisfying (29) and (30), that is

$$\theta_T^0 \cdot 1 + \boldsymbol{\theta}_T \cdot \widehat{\mathbf{S}}_T = w + \int_0^T \boldsymbol{\theta}_u \cdot d\widehat{\mathbf{S}}_u - \int_0^T e^{-ru} c_u du$$

- or, from (30)

$$e^{-rT}Z + \int_0^T e^{-ru}c_u du = w + \int_0^T \boldsymbol{\theta}_u \cdot d\widehat{\mathbf{S}}_u \qquad (63)$$

- It is easy to show that the LHS of (11) has finite expectation under Q.
- Since $\widehat{\mathbf{S}}$ is a martingale under Q we have that $M_t = w + \int_0^t \boldsymbol{\theta}_s d\widehat{\mathbf{S}}$ is a local martingale. It is also non-negative by (29), and hence it is a supermartingale under Q (for this, see TN 0).
- This implies

$$E^Q\left(w + \int_0^T \boldsymbol{\theta}_u d\widehat{\mathbf{S}}_u\right) \leq w$$

- Hence, from (11) we have

$$\begin{aligned} E^Q\left(e^{-rT}Z + \int_0^T e^{-ru}c_u du\right) &= E^Q\left(w + \int_0^T \boldsymbol{\theta}_u \cdot d\widehat{\mathbf{S}}_u\right) \\ &\leq w \end{aligned}$$

- IF. Suppose (31) is satisfied by $(c, Z) \in \widehat{D}$. Define the $Q-$martingale by

$$M_t = E_t^Q\left(e^{-rT}Z + \int_0^T e^{-rt}c_t dt\right) \qquad (64)$$

- where we assume $M_0 = w$ without loss of generality.
- Using the martingale representation theorem, we can find a d-dimensional process $\boldsymbol{\eta}$ such that

$$M_t = w + \int_0^t \boldsymbol{\eta}_s d\widehat{\mathbf{B}}_s$$

- where $\widehat{\mathbf{B}}_t = \mathbf{B}_t + \boldsymbol{\nu} t$ is a Brownian motion under Q (see Girsanov's Theorem).
- Notice that using the definition of $\widehat{\mathbf{B}}$ and $\boldsymbol{\nu} = \boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}_d)$ we have immediately

$$d\widehat{S}_t^i = \widehat{S}_t^i \boldsymbol{\sigma}^i d\widehat{\mathbf{B}}_t$$

- Choose now a process $\{\boldsymbol{\theta}_t\}$ that satisfies

$$\left(\boldsymbol{\theta}_t \odot \widehat{\mathbf{S}}_t\right) \cdot \boldsymbol{\sigma} = \boldsymbol{\eta}_t$$

- where $\odot$ denotes the element-by-element product. Such a process can be found because $\boldsymbol{\sigma}$ is invertible.
- Hence

$$\begin{aligned} M_t &= w + \int_0^t \boldsymbol{\eta}_u d\widehat{\mathbf{B}}_u = w + \sum_{i=1}^d \int_0^t \boldsymbol{\theta}_u^i \widehat{\mathbf{S}}_u^i \boldsymbol{\sigma}^i d\widehat{\mathbf{B}}_u \\ &= w + \int_0^t \boldsymbol{\theta}_u d\widehat{\mathbf{S}}_u \qquad (65) \end{aligned}$$

- Also, let the investment in bonds be the residual:

$$\theta_t^0 = M_t - \boldsymbol{\theta}_t \cdot \widehat{\mathbf{S}}_t - \int_0^t e^{-ru} c_u du$$

- Clearly, we have

$$\begin{aligned} \theta_t^0 + \boldsymbol{\theta}_t \cdot \widehat{\mathbf{S}}_t &= M_t - \int_0^t e^{-ru} c_u du \\ &= E_t^Q \left(e^{-rT} Z + \int_t^T e^{-ru} c_u du \right) \geq 0 \end{aligned}$$

- where we used (64) for M_t. In addition, using instead (65)

$$\begin{aligned} \theta_t^0 + \boldsymbol{\theta}_t \cdot \widehat{\mathbf{S}}_t &= M_t - \int_0^t e^{-ru} c_u du \\ &= w + \int_0^t \boldsymbol{\theta}_u d\widehat{\mathbf{S}}_u - \int_0^t e^{-ru} c_u du \end{aligned}$$

- and

$$\begin{aligned} \theta_T^0 + \boldsymbol{\theta}_T \cdot \widehat{\mathbf{S}}_T &= M_T - \int_0^T e^{-ru} c_u du \\ &= e^{-rT} Z + \int_0^T e^{-rt} c_t dt - \int_0^T e^{-ru} c_u du \\ &= e^{-rT} Z \end{aligned}$$

- where the second equality stems again from the definition of M_t in (64).

- This shows that (c, Z) is budget feasible with respect to $\widehat{\mathbf{S}}$, which implies the one with respect to $\mathbf{S}$, concluding the proof.$\square$