Teaching Notes #4 (Addendum)

Portfolio Selection with recursive preferences and time varying expected return[1]

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- Consider again the examples in TN 1 and TN 3, Addendum.
- Consider the case with only one asset, with stock dynamics

$$dS_t/S_t = \mu_t dt + \sigma_S dB_t$$

and

$$d\mu_t = (A_0 + A_1\mu_t)\, dt + \sigma_{\mu,t} dB_t$$

- We are in complete markets and $\lambda_t = \mu_t - r$. Of course

$$d\lambda_t = \left(\widehat{A}_0 + A_1\lambda_t\right) dt + \sigma_{\lambda,t} dB_t$$

where

$$\widehat{A}_0 = (A_0 + A_1 r) \quad \text{and} \quad \sigma_{\lambda,t} = \sigma_{\mu,t}$$

[1] This teaching notes partly draw on Campbell, Chacko, Rodriguez and Viceira "Strategic Asset Allocation in a Continuous Time VAR model", JEDC, 2004.

- We may also think of this setting as one where the true expected return $\overline{\mu}$ is constant but investors learn about it. In this case $\sigma_{\mu,t}$ decreases to zero according to a specific deterministic path (see TN 3).
- We consider λ_t our state variable
- For a *normalized aggregator*, the Bellman equation turns out to be

$$\sup_{(c,\vartheta)} \mathcal{D}^{(c,\vartheta)} J(\lambda, w, t) + \overline{f}(c, J(\lambda, w, t)) = 0 \qquad (1)$$

- where

$$\begin{aligned} \mathcal{D}^{(c,\vartheta)} J(\lambda, w, t) &= J_t + J_\lambda \left(\widehat{A}_0 + A_1 \lambda_t\right) + J_w \left(w\vartheta\lambda_t + wr - c\right) \\ &\quad + \frac{1}{2}\left(\sigma_{\lambda,t}^2 J_{\lambda\lambda} + 2w\vartheta J_{w\lambda}\sigma_{\lambda,t}\sigma_S + \sigma_S^2 w^2 \vartheta^2 J_{ww}\right) \end{aligned}$$

- The FOC with respect to consumption and portfolio holdings are

$$\begin{aligned} J_w &= \overline{f}_c \\ \vartheta &= -\frac{J_w w \lambda_t + w J_{w\lambda}\sigma_\lambda \sigma_S}{w^2 \sigma_S^2 J_{ww}} \end{aligned}$$

- For KP, we have

$$\overline{f}(c, J) = \frac{\phi}{\rho}\frac{c^\rho - (\alpha J)^{\frac{\rho}{\alpha}}}{(\alpha J)^{\left(\frac{\rho}{\alpha}-1\right)}} = \frac{\phi}{\rho}\alpha J \left(\left(\frac{c}{(\alpha J)^{\frac{1}{\alpha}}}\right)^\rho - 1\right) \qquad (2)$$

- Recall that we have the following interpretation of parameters

$$\rho = 1 - \frac{1}{\psi}$$
$$\alpha = 1 - \gamma$$

where γ is a risk aversion parameter, and ψ is the elasticity of intertemporal substitution.

- For these teaching notes, I consider the special case where $\psi = 1$, that is $\rho = 0$. This corresponds to the log utility in the power utility ($\alpha = \rho$).

- The reason is that in this case, we obtain a closed form formula for both consumption and portfolio allocation rules.

- Campbell and Viceira (2002), and Campbell et al. (2004) contain the methodology and results to obtain approximate solutions for the case $\psi \neq 1$.

- In the case $\rho = 0$ the aggregator is

$$f(c, J) = \phi \alpha J \left[\log(c) - \frac{1}{\alpha} \log(\alpha J) \right]$$

- Conjecture

$$J = \psi(\lambda, t) \frac{w^{\alpha}}{\alpha}$$

- Then

$$J_w = \psi w^{\alpha - 1} = \alpha J w^{-1}$$
$$J_{ww} = \psi(\alpha - 1) w^{\alpha - 2} = (\alpha - 1) \alpha J w^{-2}$$
$$J_t = \psi_t \frac{w^{\alpha}}{\alpha} = \frac{\psi_t}{\psi} J$$

$$
\begin{aligned}
J_\lambda &= \psi_\lambda \frac{w^\alpha}{\alpha} = \frac{\psi_\lambda}{\psi} J \\
J_{\lambda\lambda} &= \psi_{\lambda\lambda} \frac{w^\alpha}{\alpha} = \frac{\psi_{\lambda\lambda}}{\psi} J \\
J_{\lambda w} &= \psi_\lambda w^{\alpha-1} = \frac{\psi_\lambda}{\psi} \alpha J w^{-1}
\end{aligned}
$$

- From FOC with respect to consumption

$$f_c = \phi \alpha J c^{-1} = J_w$$

- we obtain

$$c = \phi \frac{\alpha J}{J_w} = \phi w$$

- That is, a unit elasticity of intertemporal substitution implies a constant consumption/wealth ratio.
  - An increase in expected return has two effects: it increases future expected consumption, thereby increasing the desire to consume more today to smooth out consumption.
  - However, it also increases the desire to invest more money in assets, and hence to consume less today.
  - When EIS = 1, the two effects exactly balance each other.
- In addition, the FOC with respect to ϑ become

$$\vartheta_t = \frac{\lambda_t}{\sigma_S^2 (1-\alpha)} + \frac{\psi_\lambda / \psi}{(1-\alpha)} \frac{\sigma_{\lambda,t}}{\sigma_S}$$

- We have the usual two terms: Myopic demand plus hedging demand.

- Substitute everything in the bellman equation, and we find

$$
\begin{aligned}
0 &= \phi\alpha J\left[\log(\phi)-\frac{1}{\alpha}\log(\psi(\lambda,t))\right] \\
&\quad+\frac{\psi_t}{\psi}J+\frac{\psi_\lambda}{\psi}J\left(\widehat{A}_0+A_1\lambda_t\right) \\
&\quad+\alpha J\left(\vartheta_t\lambda_t+r-\phi\right) \\
&\quad+\frac{1}{2}\left(\sigma_{\lambda,t}^2\frac{\psi_{\lambda\lambda}}{\psi}J+2\vartheta\frac{\psi_\lambda}{\psi}\alpha J\sigma_{\lambda,t}\sigma_S+\sigma_S^2\vartheta^2(\alpha-1)\alpha J\right)
\end{aligned}
$$

- After some algebra we find the PDE:

$$
\begin{aligned}
0 &= \phi\alpha\log(\phi)-\phi\log(\psi)+\frac{\psi_t}{\psi}+\frac{\psi_\lambda}{\psi}\left(\widehat{A}_0+A_1\lambda_t\right) \\
&\quad+\frac{1}{2}\alpha\frac{\lambda_t^2}{\sigma_S^2(1-\alpha)}+\alpha r-\alpha\phi+\frac{1}{2}\sigma_{\lambda,t}^2\frac{\psi_{\lambda\lambda}}{\psi}+\frac{\lambda_t\frac{\psi_\lambda}{\psi}\alpha\sigma_{\lambda,t}}{\sigma_S(1-\alpha)}+ \\
&\quad+\frac{1}{2}\frac{\left(\frac{\psi_\lambda}{\psi}\right)^2\sigma_{\lambda,t}^2\alpha}{(1-\alpha)}
\end{aligned}
$$

- Conjecture that the solution to the PDE is given by

$$\psi(\lambda,t)=e^{a_0(t;T)+a_1(t;T)\lambda_t+a_2(t;T)\lambda_t^2}$$

- This implies

$$
\begin{aligned}
\psi_t &= \left(a_0'+a_1'\lambda_t+a_2'\lambda_t^2\right)\psi \\
\psi_\lambda &= \left(a_1+2a_2\lambda_t\right)\psi \\
\psi_{\lambda\lambda} &= \left(2a_2+\left(a_1+2a_2\lambda_t\right)^2\right)\psi
\end{aligned}
$$

- Substitute to find

$$0 = \phi\alpha\log(\phi)-\phi\left(a_0+a_1\lambda_t+a_2\lambda_t^2\right)$$

$$+\left(a_0'+a_1'\lambda_t+a_2'\lambda_t^2\right)+\left(a_1+2a_2\lambda_t\right)\left(\widehat{A}_0+A_1\lambda_t\right)$$
$$+\frac{1}{2}\alpha\frac{\lambda_t^2}{\sigma_S^2\left(1-\alpha\right)}+\boldsymbol{\alpha}r-\alpha\phi+\frac{1}{2}\sigma_{\lambda,t}^2\left(2a_2+\left(a_1+2a_2\lambda_t\right)^2\right)$$
$$+\frac{\lambda_t\alpha\sigma_{\lambda,t}}{\sigma_S\left(1-\alpha\right)}\left(a_1+2a_2\lambda_t\right)+$$
$$+\frac{1}{2}\frac{\sigma_{\lambda,t}^2\alpha}{\left(1-\alpha\right)}\left(a_1+2a_2\lambda_t\right)^2$$

- Collect terms

$$\begin{aligned}0 &= \left(a_1'-\phi a_1+a_1A_1+2a_2\widehat{A}_0++\sigma_{\lambda,t}^2a_1a_2+\frac{\alpha\sigma_{\lambda,t}}{\sigma_S\left(1-\alpha\right)}a_1+a_1\frac{\sigma_{\lambda,t}^2\alpha}{\left(1-\alpha\right)}a_1a_2\right)\lambda_t\\ &\quad+\left(a_2'-\phi a_2+2a_2A_1+\frac{\frac{1}{2}\alpha}{\sigma_S^2\left(1-\alpha\right)}+2\sigma_{\lambda,t}^2a_2^2+\frac{\alpha\sigma_{\lambda,t}}{\sigma_S\left(1-\alpha\right)}2a_2+2\frac{\sigma_{\lambda,t}^2\alpha}{\left(1-\alpha\right)}a_2^2\right)\lambda_t^2\\ &\quad+a_0'+\phi\alpha\log\left(\phi\right)-\phi a_0+a_1\widehat{A}_0+\boldsymbol{\alpha}r-\alpha\phi+\sigma_{\lambda,t}^2a_2+\frac{1}{2}\sigma_{\lambda,t}^2a_1^2+\frac{1}{2}\frac{\sigma_{\lambda,t}^2\alpha}{\left(1-\alpha\right)}a_1^2+\end{aligned}$$

- and obtain the recursive system

$$\begin{aligned}a_2'-\phi a_2+2a_2A_1+\frac{\frac{1}{2}\alpha}{\sigma_S^2\left(1-\alpha\right)}+2\sigma_{\lambda,t}^2a_2^2+\frac{\alpha\sigma_{\lambda,t}}{\sigma_S\left(1-\alpha\right)}2a_2+2\frac{\sigma_{\lambda,t}^2\alpha}{\left(1-\alpha\right)}a_2^2 &= 0\\ a_1'-\phi a_1+a_1A_1+2a_2\widehat{A}_0++\sigma_{\lambda,t}^2a_1a_2+\frac{\alpha\sigma_{\lambda,t}}{\sigma_S\left(1-\alpha\right)}a_1+a_1\frac{\sigma_{\lambda,t}^2\alpha}{\left(1-\alpha\right)}a_1a_2 &= 0\\ a_0'+\phi\alpha\log\left(\phi\right)-\phi a_0+a_1\widehat{A}_0+\boldsymbol{\alpha}r-\alpha\phi+\sigma_{\lambda,t}^2a_2+\frac{1}{2}\sigma_{\lambda,t}^2a_1^2+\frac{1}{2}\frac{\sigma_{\lambda,t}^2\alpha}{\left(1-\alpha\right)}a_1^2 &= 0\end{aligned}$$

- **Calibration**
- We use the learning model studied discussed in TN 3 Addendum.
- Recall that the parameters are as follows:

Parameter Choice

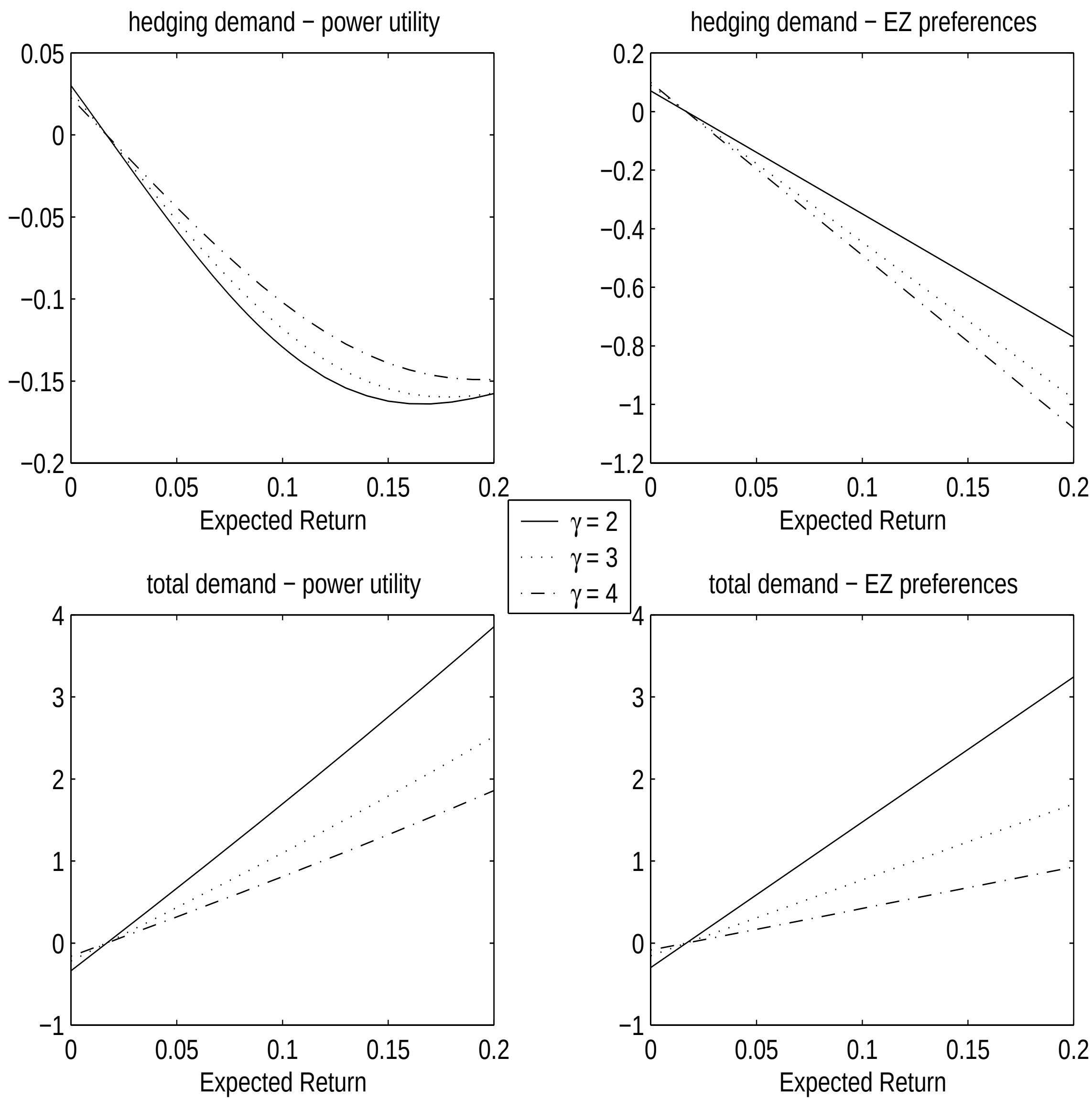
Rate of time preference ϕ	0.0624
Risk free rate r	0.0168
Volatility of stock prices σ_S	0.1510
γ	3
T	25
Uncertainty $\sqrt{q_0}$	2%

- The only difference from TN 3 are the assumptions for the base case parameters. Here, we assume $\gamma = 3$ instead of 5, $T = 25$ instead of 50, and uncertainty $\sqrt{q_0} = 2\%$ instead of 3%.

- The reason is that the effects are all stronger with EZ preferences and EIS = 1, and so we do not need extreme values to obtain reasonable effects.

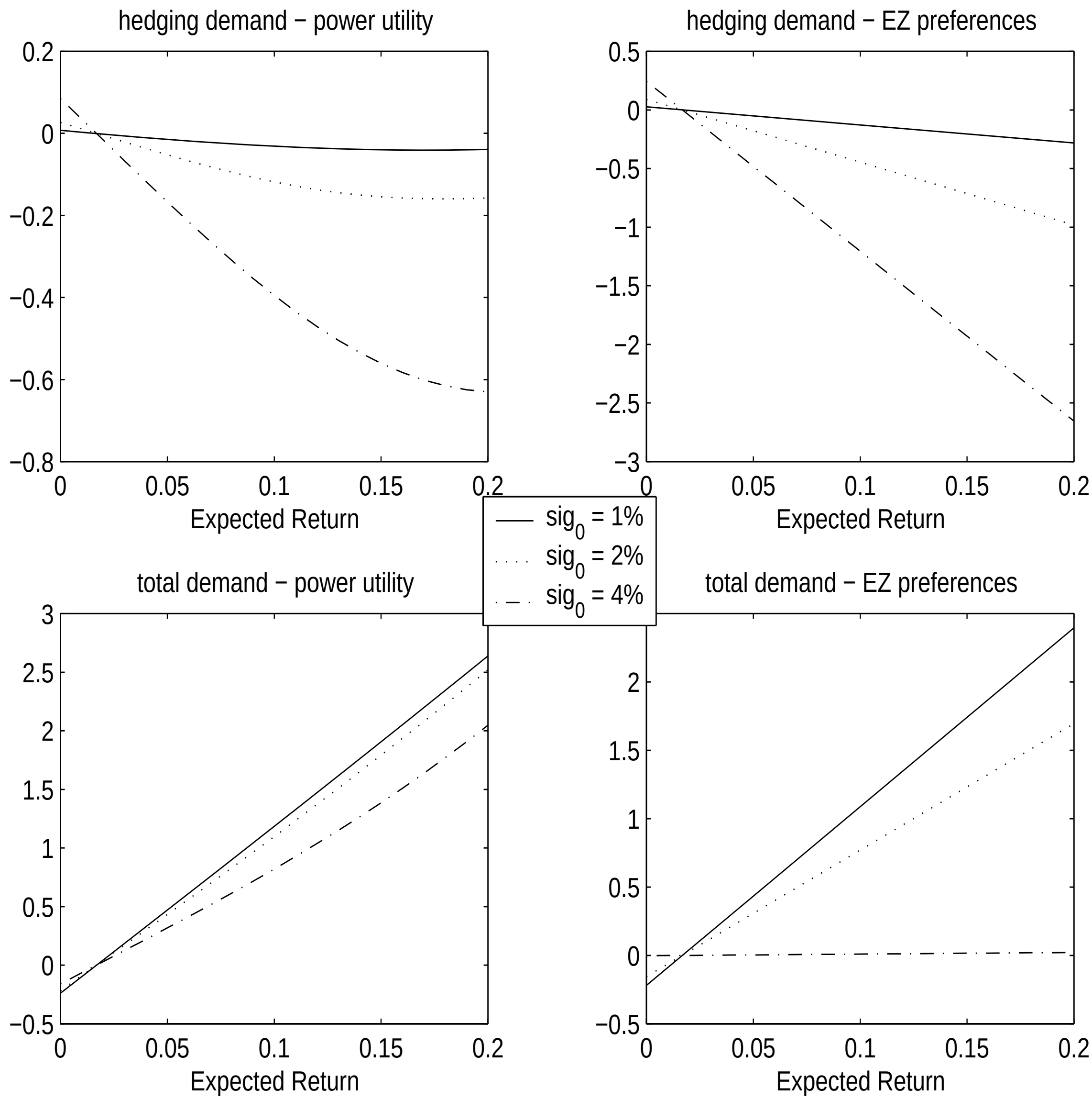
- Figure 1 plots the hedging demand and total demand for various γ, plotted against expected returns.

Figure 1: Portfolio Allocation for various γ



- Figure 2 plots the hedging demand and total demand for various uncertainty levels $\sigma_0 = \sqrt{q_t}$, plotted against expected returns.

Figure 2: Portfolio Allocation for various uncertainty levels



- For given parameter γ, the hedging demand effect for EZ preferences is stronger than for power utility.
- Recall the intuition for the existence of hedging demand with learning:
  - Bad news about stock returns (a negative shock) is double bad news, as it also implies that expected *future* returns are low.
  - The investor, forecasting this correlation between returns and expected return, decreases *today* the demand of stock compared to the myopic case.
  - When EIS ¡ 1, however, part of the bad news is amortized through consumption smoothing: if the investor receive a bad news, he/she will adapt not only the asset allocation, but also consumption.
  - Instead, EIS = 1 implies that $C/W = \phi =$ constant, and thus the agent cannot (i.e. is not willing to) amortize bad shocks to changes in consumption.
  - It then decrease substantially the demand for stocks to avoid to be over-exposed to return shocks.