Topics in Dynamic Asset Pricing

Lecture Notes 6:
Structural Credit Risk Models

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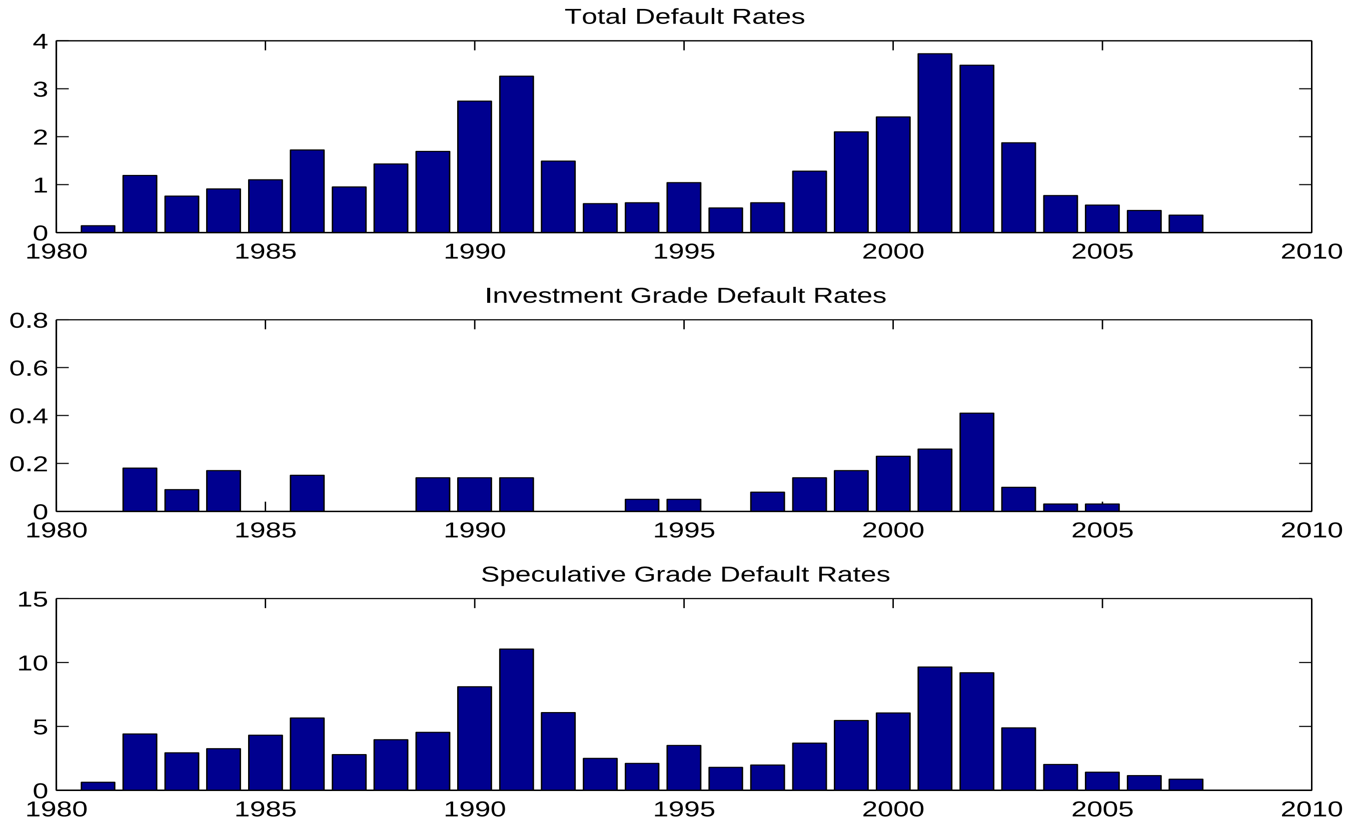
Introduction

- Default rates vary both over time and across "types" of corporations.
- Figure 1 shows global default rates from 1980 to 2003 across various bond qualities.

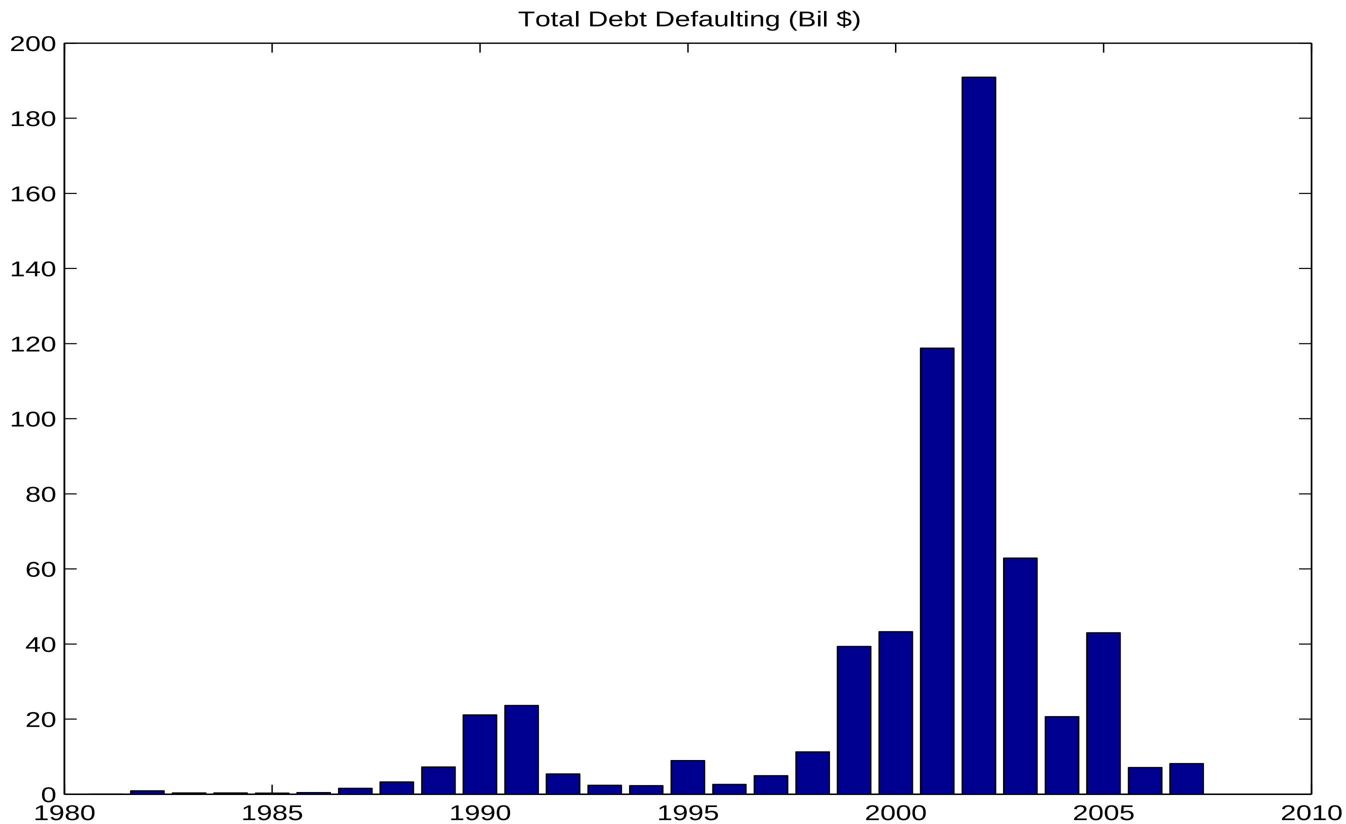
Figure 1: Global Default Rates



(source: Standard and Poor)

- The total amount of debt subject to default is large, and it varies over time. It reached a peak in 2002 of over 180 Billion dollar.

Figure 2: Total Debt Defaulting

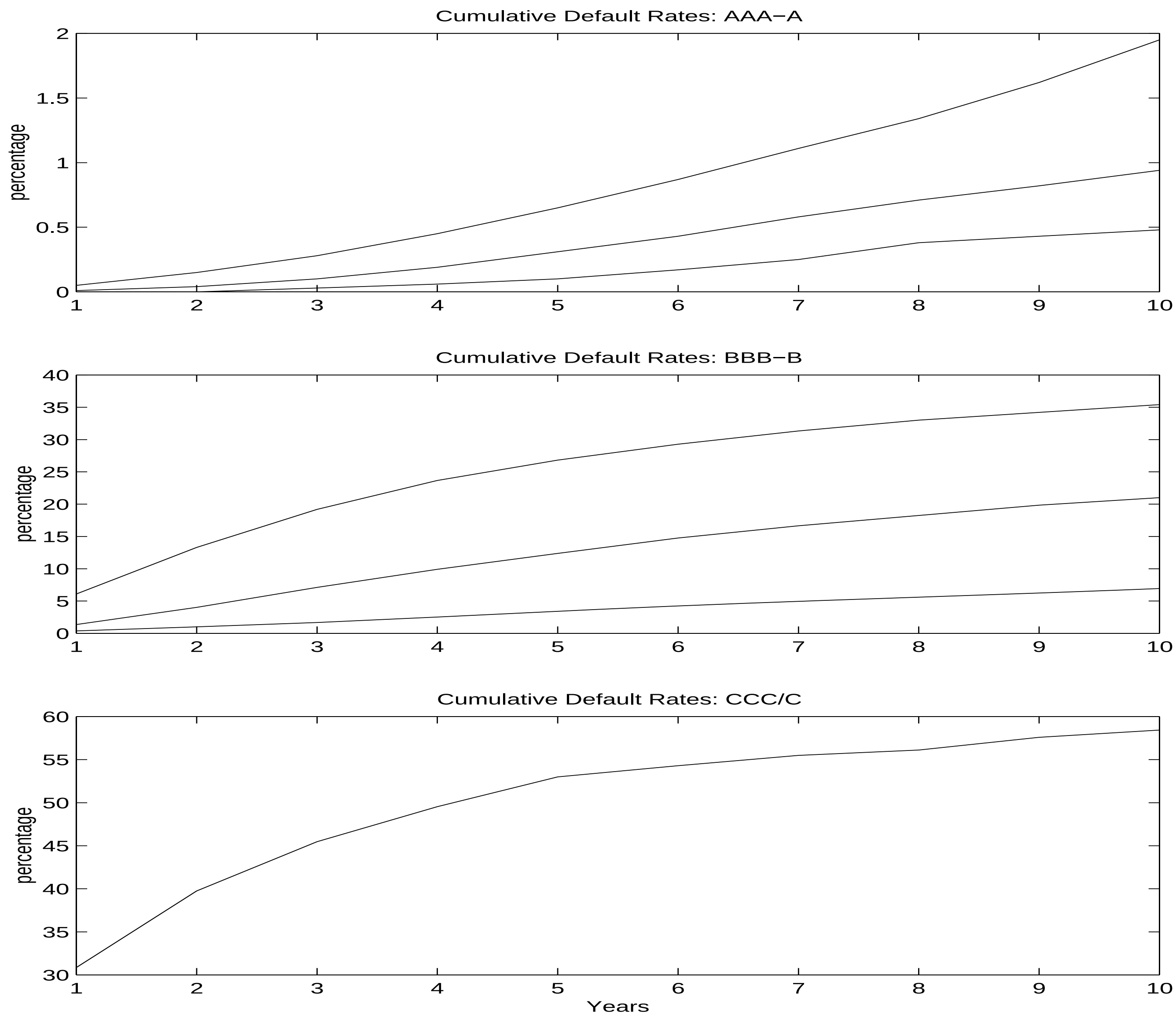


(source: Standard and Poor)

- The probability of default varies across firms. Credit rating agencies provide average default rates for various catergories.
  - Figure 3 show the cumulative average default rates for firms that in year 0 were rated AAA-A (Top Panel), BBB-B (Medium Panel), or CCC (Bottom Panel).

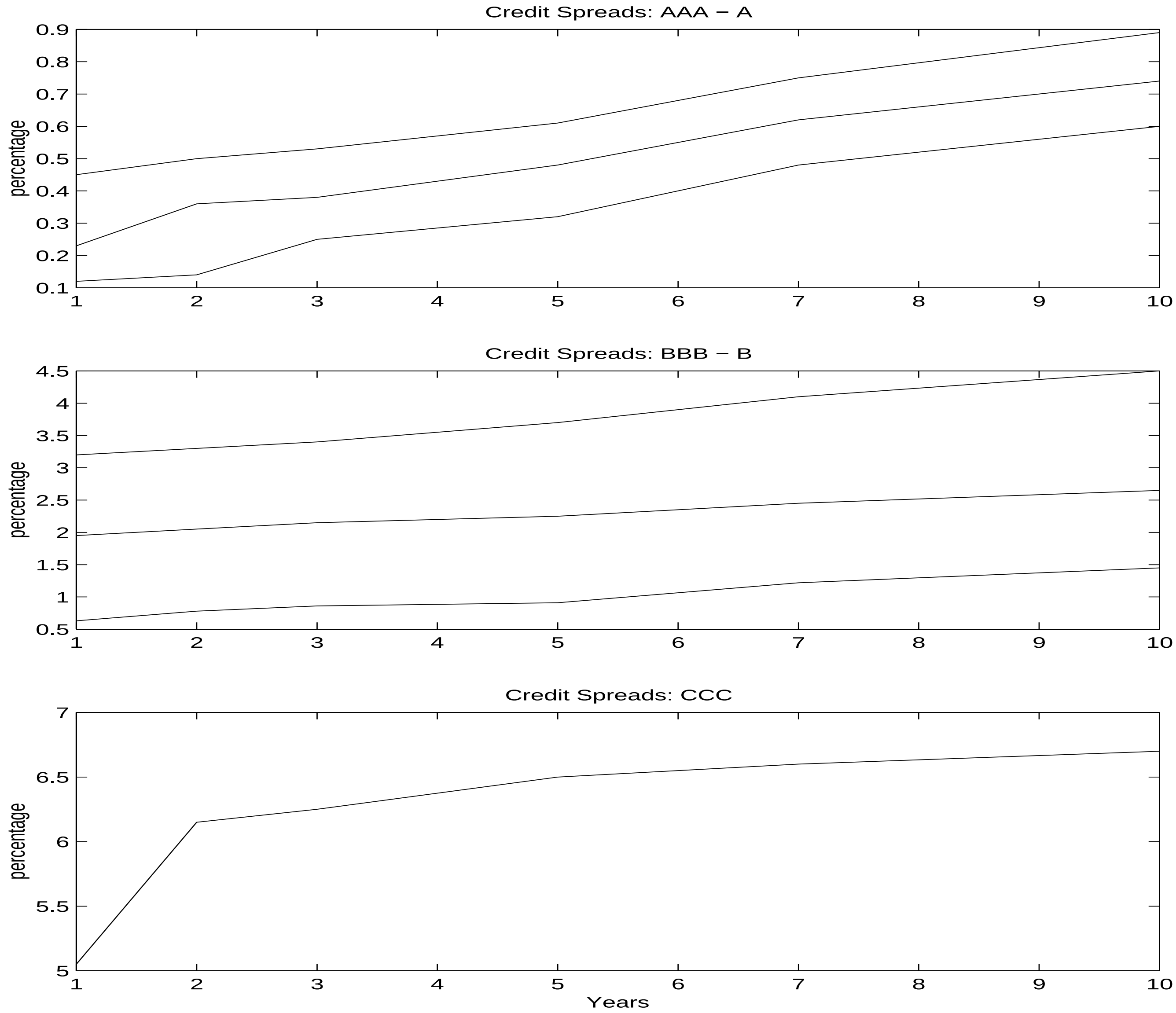
- These various default probabilities are reflected in credit spreads, that is, the difference in average yield between bonds in the various credit classes, and treasuries.
  - Figure 4 plots the credit spreads with respect to maturity. Figure 5 and 6 plot their time variation.

Figure 3: Cumulative Default Rates



(source: Standard and Poor)

Figure 4: Credit Spreads



(source: Reuter)

Figure 5: Corporate yields over time

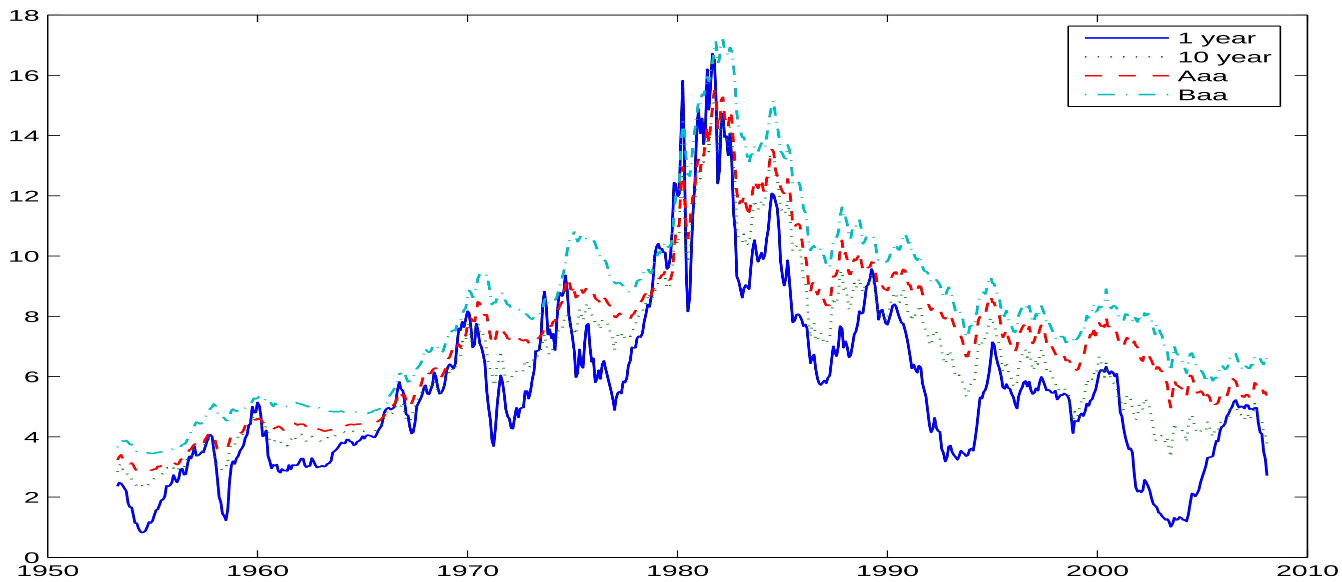
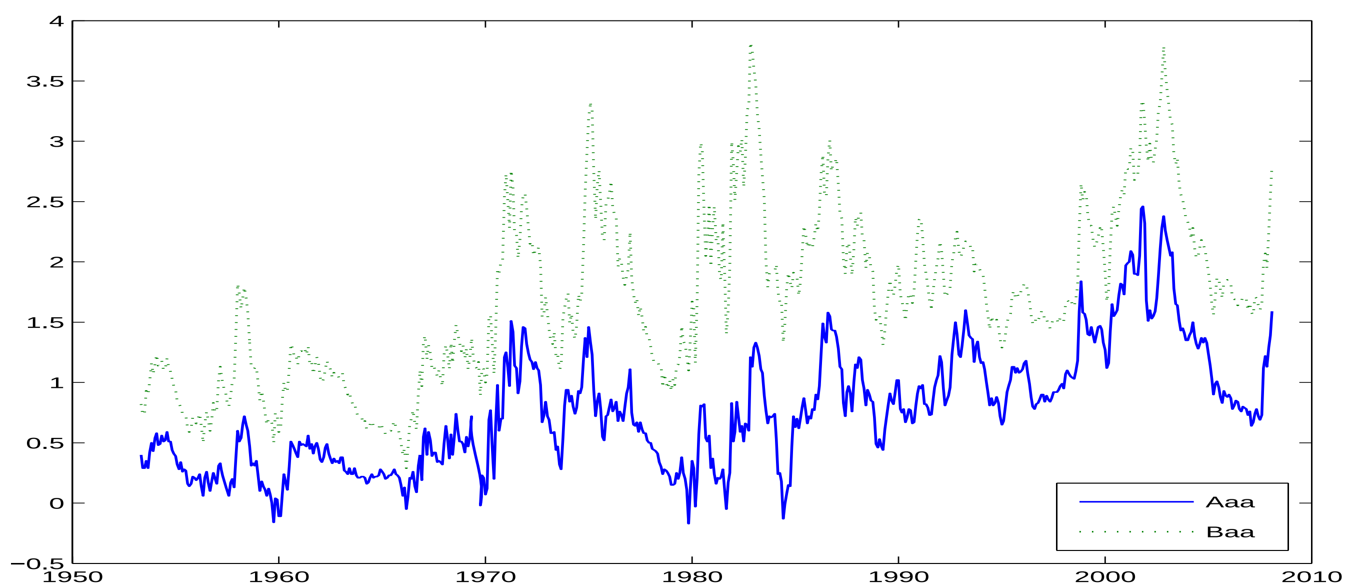


Figure 6: Corporate spreads over time



Structural Approach to Default Risk

- Structural default approach = generalization of Merton (1974) model
- Merton (1974): Asset value follows lognormal process $dV = \mu V dt + \sigma V dW_1$
- Debt = zero coupon bond maturing at T: $\Longrightarrow$ Payoff $= F(V) = V - max(V - D, 0)$
- Contingent claim analysis: Let $P(A, t; T) =$ value of the claim to payoff above.
  - Consider a portfolio $\Pi = P(V, t; T) - \Delta V$. Ito's Lemma implies

$$d\Pi = \left(P_t + \frac{1}{2}P_{VV}V^2\sigma^2\right) dt + (P_V - \Delta)\, dV$$

  - Choose Δ to eliminate "dV": $\Longrightarrow \Delta = P_V$. Since resulting $d\Pi$ is locally riskless, impose $d\Pi = r\Pi dt$, substitute, and obtain

$$rP = P_t + \frac{1}{2}P_{VV}V^2\sigma^2 + rVP_V$$

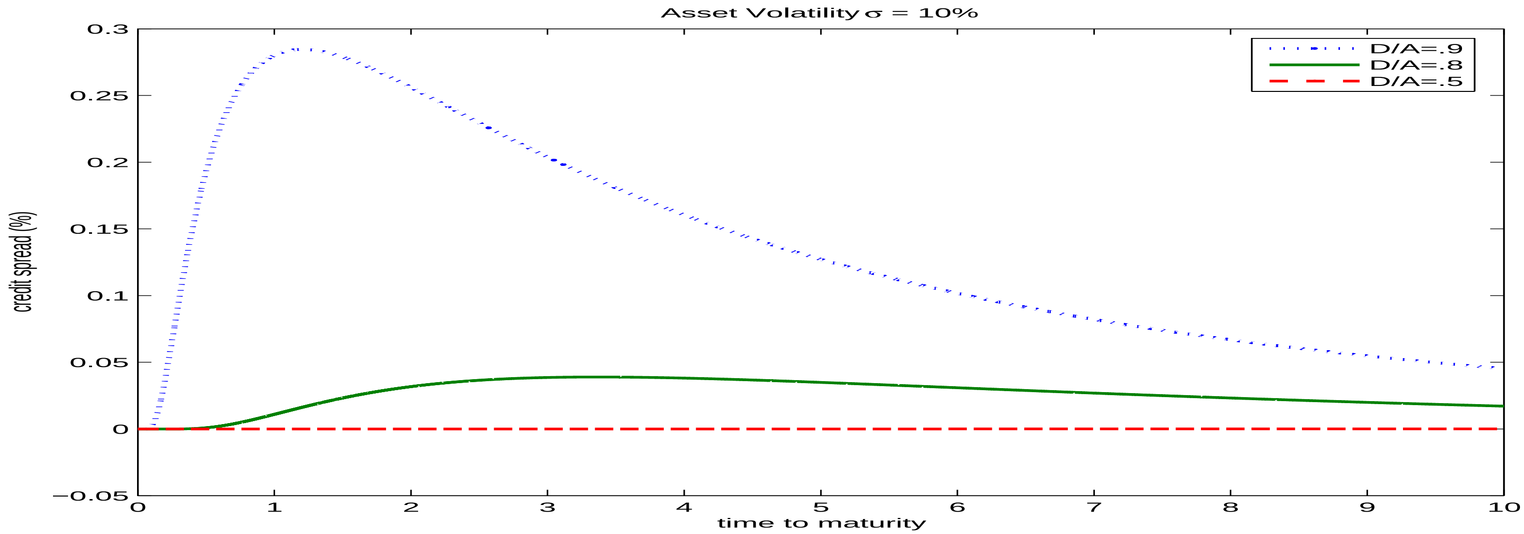
- Feynman Kac solution to the value of debt

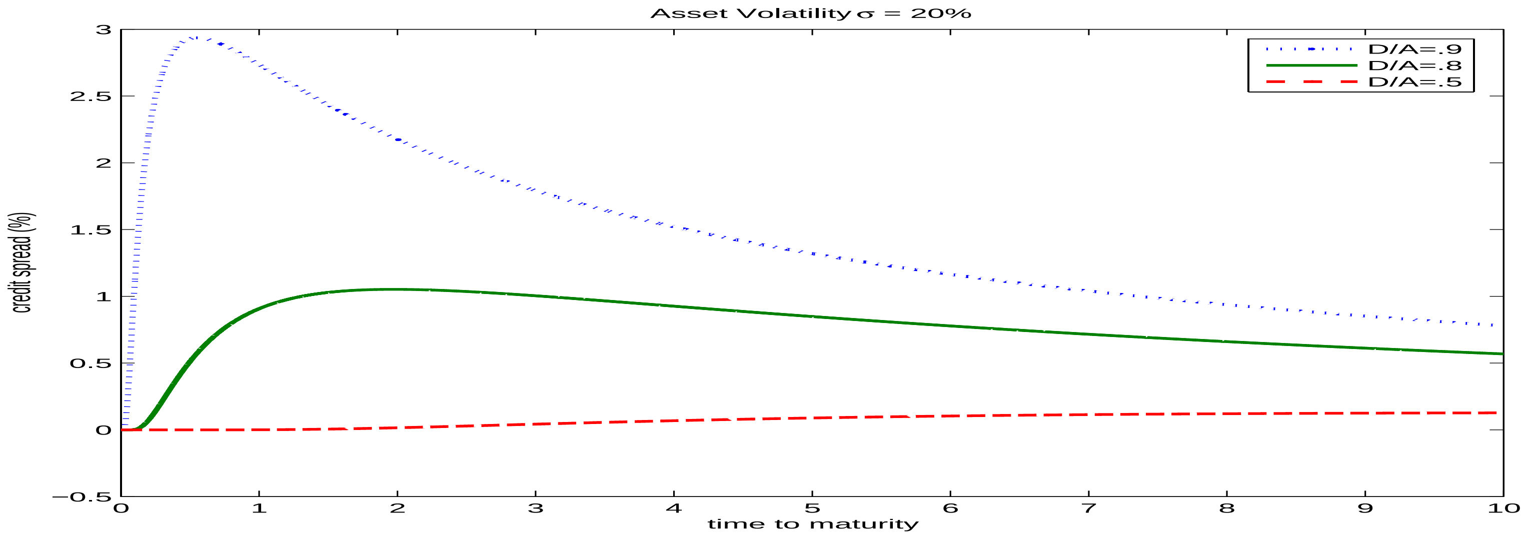
$$P(V, t; T) = E^Q\left[e^{-r(T-t)}F(V)\right]$$

where expectation is taken under the risk neutral dynamics

$$dV = rV dt + \sigma V dW$$

Credit Spreads Implied by Merton Model





- Issues: (A) they are too low; (B) they converge to zero for $T \longrightarrow 0$

Equity Market Volatility under Merton Model

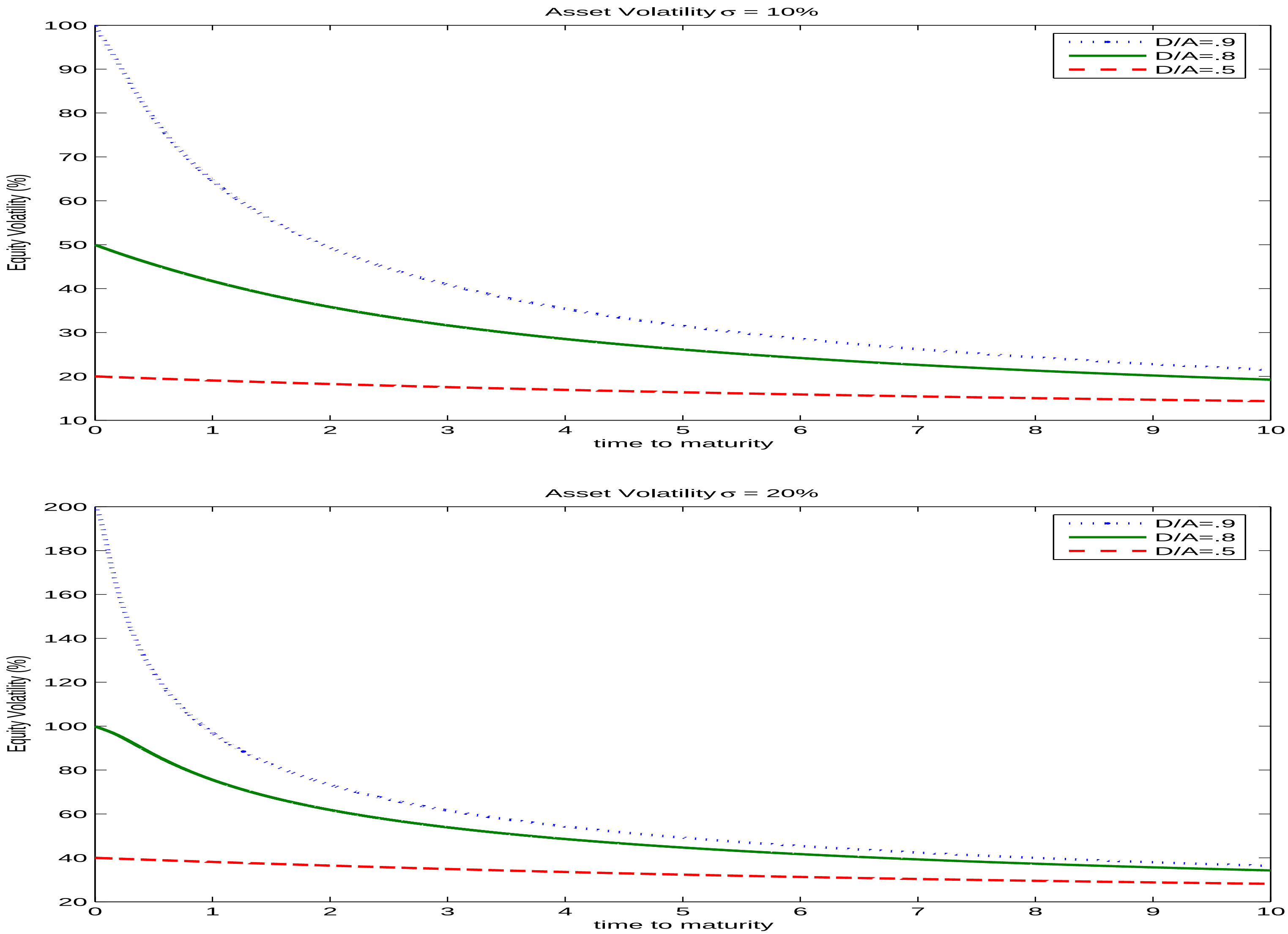
- σ is not observable. So, how do we know if $\sigma = 20\%$ is too high?
- Look at equity volatility. Equity is a call option, thus under the risk neutral dynamics

$$dE = rEdt + \sigma_E dW$$

- where

$$\sigma_E = \frac{N(d_1)V\sigma}{N(d_1)V\sigma - De^{-r(T-t)}N(d_2)}$$

Equity Market Volatility under Merton Model



Longstaff and Schwartz Model

1. Vasicek interest rate model: $dr = (\xi - \beta r)dt + \nu dW_2$
   - where $dW_1 dW_2 = \rho dt$
2. As in Black and Cox (1976), default occurs when $V_t \leq V^*$
3. In case of default, bond holders get fraction $1 - \omega$ of *face value* of debt at T.

- Bond price depends on V and r: $P(V, r, t; T)$. The Fundamental Pricing Equation is

$$rP = P_t + \frac{1}{2}P_{VV}V^2\sigma^2 + rVP_V + P_r(\alpha - \beta r) + \frac{1}{2}P_{rr}\nu^2 + P_{rV}V\nu\sigma\rho$$

- with boundary condition $P(V, r, T; T) = 1 - \omega I_{\tau \leq T}$ where $\tau = inf\{t : V_t < V^*\}$ = time of default. (Above $\alpha = \xi -$ market price of risk.)

- One can apply FK theorem and find

$$P(V, r, t; T) = E^*\left[e^{-\int_t^T r_u du}(1 - \omega I_{\tau \leq T})\right]$$

under the risk neutral dynamics

$$\begin{aligned} d\ln V &= (r - \frac{1}{2}\sigma^2)dt + \sigma dW_1 \\ dr &= (\alpha - \beta r - \nu^2 B(t; T))dt + \nu dW_2 \end{aligned}$$

- This is still hard to evaluate.

Longstaff and Schwartz Model

- **Proposition:** Let $X = V_t/V^*$. Then $P(X, r, t; T) = Z(r, t; T)(1 - \omega Q(X, r, t; T))$ where $Z(r, t; T) =$ default free bond, and $Q(X, t; T)$ satisfies

  $$0 = Q_t + \frac{1}{2}Q_{XX}X^2\sigma^2 + (r - \sigma\rho\nu B(t;T))XQ_X + Q_r(a - \beta r - \nu^2 B(t;T)) + \frac{1}{2}Q_{rr}\nu^2 + Q_{rX}X\sigma\nu\rho$$

  with final condition $Q(X, r, T; T) = I_{\tau \leq T}$.

- Above, $B(t;T)$ comes from the Vasicek Bond Pricing Formula $Z(r, t; T) = e^{A(t;T) - B(t;T)r}$, where

  $$B(t;T) = \frac{1 - e^{-\beta(T-t)}}{\beta}$$

- If we apply FK theorem to previous equation, we find $Q(X, r, t; T) = E_f^*[I_{\tau \leq T}]$ under the dynamics

  $$\begin{aligned} d\ln X &= (r - \frac{1}{2}\sigma^2 - \sigma\nu\rho B(t;T))dt + \sigma dW_1 \\ dr &= (a - \beta r - \nu^2 B(t;T))dt + \nu dW_2 \end{aligned}$$

- This is called the **Forward Risk Neutral Measure**. It results from a change of numeraire from dollars into units of a zero coupon bond $Z(r, t; T)$.
- $Q(X, r, t; T) =$ first passage time probability under the forward risk neutral dynamics.
- Solution in (almost) closed form is known. See Paper or Huang and Huang (2003).

Table 3: Effects of Stochastic Interest Rates

Panel A: Maturity = 10 years

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	Target			*Implied*				
Credit Rating	Leverage Ratio (%)	Equity Prem (%)	Cum. Default Prob. (%)	Asset Vol. (%)	Asset Risk Prem (%)	**Calc Credit Spread** (bps)	Avg Yield Spread (bps)	% of Spread due to Default
Aaa	13.1	5.38	0.77	31.5	4.99	6.0	63	9.6
Aa	21.2	5.60	0.99	27.5	4.95	8.6	91	9.4
A	32.0	5.99	1.55	24.5	4.94	14.5	123	11.8
Baa	43.3	6.55	4.39	24.7	5.05	38.6	194	19.9
Ba	53.5	7.30	20.63	31.3	5.47	153.9	320	48.1
B	65.7	8.76	43.91	38.4	6.42	341.9	470	72.8

Panel B: Maturity = 4 years

Aaa	13.1	5.38	0.04	36.6	4.96	0.8	55	1.5
Aa	21.2	5.60	0.23	34.8	4.91	4.6	65	7.0
A	32.0	5.99	0.35	30.0	4.88	7.5	96	7.8
Baa	43.3	6.55	1.24	29.1	4.94	25.4	158	16.1
Ba	53.5	7.30	8.51	34.3	5.26	149.2	320	46.6
B	65.7	8.76	23.32	39.3	6.15	406.0	470	86.4

Table 3 reports calibration results and calculated credit yield spreads for two maturities in the case of stochastic interest rates, implemented by using the Longstaff and Schwartz (1995) model. The setup is otherwise identical to that of the base case. For bonds of each credit rating and maturity, the model calibrated to match the target initial leverage ratio (column 2), equity premium (column 3), and cumulative default probability until bond maturity (column 4) is shown to have the implied asset volatility and the asset risk premium given in columns 5 and 6. The credit yield spreads calculated from such calibrated models (column 7) are also shown as a percentage (column 9) of historically observed yield spreads for the bonds (column 8). Parameter choices are as follows: the coupon rate is 8.162%; the asset payout ratio $\delta = 6\%$; the default boundary V^* equals 60% of the total face value of the firms' bonds outstanding; and the recovery rate given default is fixed at 51.31%. The parameters for the interest rate process are $r_0 = 8\%, \kappa_r = 0.226, \theta = 0.113, \sigma_r = 4.68\%$, and $\pi_0^r = -0.248$. The correlation coefficient ρ_{rV} is chosen to be -0.25.

Source: Huang Huang 2003

Collin-Dufresne - Goldstein Model

- Capital structure of firms change over time, perhaps due to frictions.
  - Firms issue more debt when V_t is too high, to push it back to a target.
  - Firms hesitate to replace maturing debt when V_t is too low.

- CDG Model: assume that threshold V_t^* is also time varying, and moves according to

$$d\ln(V_t^*) = \lambda[\ln(V_t) - k - \ln(V_t^*) - \phi(r_t - \theta)]dt$$

- That is, if $r_t = \theta$
  - if $\ln(V_t^*) + k < \ln(V_t)$, management acts to increase threshold V_t^*, and viceversa.
- when $r_t > \theta$
  - threshold tends to decrease, according with the intuition / evidence that firms do not issue much debt when rates are very high.
- Note that state variable $\ln X_t = \ln(V_t) - \ln(V_t^*)$ and r_t still follow a joint gaussian proces in the forward risk neutral dynamics.
- The bond pricing formula is the same as before.

Table 6: Mean-Reverting Leverage Ratios with the Collin-Dufresne and Goldstein Model

Panel A: Maturity = 10 years

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	Target			*Implied*				
Credit Rating	Leverage Ratio (%)	Equity Prem (%)	Cum. Default Prob. (%)	Asset Vol. (%)	Asset Risk Prem (%)	**Calc Credit Spread** (bps)	Avg Yield Spread (bps)	% of Spread due to Default
Aaa	7.8	5.38	0.77	25.9	4.96	11.4	63	18.2
Aa	12.7	5.60	0.99	24.9	4.91	14.9	91	16.4
A	19.2	5.99	1.55	24.5	4.89	22.5	123	18.3
Baa	26.0	6.55	4.39	26.3	5.00	52.3	194	26.9
Ba	32.1	7.30	20.63	32.9	5.46	182.7	320	57.1
B	39.4	8.76	43.91	39.9	6.43	371.6	470	79.1

Panel B: Maturity = 4 years

Aaa	7.8	5.38	0.04	25.9	4.95	0.0	55	0.1
Aa	12.7	5.60	0.23	31.9	4.90	6.3	65	9.7
A	19.2	5.99	0.35	29.1	4.86	9.9	96	10.3
Baa	26.0	6.55	1.24	29.6	4.91	31.1	158	19.7
Ba	32.1	7.30	8.51	35.8	5.28	168.0	320	52.5
B	39.4	8.76	23.32	41.8	6.21	435.3	470	92.6

Table 6 reports calibration results and calculated credit yield spreads for two maturities using the Collin-Dufresne and Goldstein model, in which the firm adjusts its debt level such that the leverage ratio is mean-reverting. The setup is otherwise identical to that of the base case. For bonds of each credit rating and maturity, the model calibrated to match the target initial leverage ratio (column 2), equity premium (column 3), and cumulative default probability until bond maturity (column 4) is shown to have the implied asset volatility and the asset risk premium given in columns 5 and 6. The credit yield spreads calculated from such calibrated models (column 7) are also shown as a percentage (column 9) of historically observed yield spreads for the bonds (column 8). Parameter choices are as follows: the coupon rate is 8.13%; the asset payout ratio $\delta = 6\%$; the interest rate is constant at 8%; the recovery rate given default is fixed at 51.31%; the mean-reversion coefficient of the leverage ratio $\kappa_\ell = 0.2$; and the long-term mean leverage ratio is 38%.

Source: Huang Huang 2003

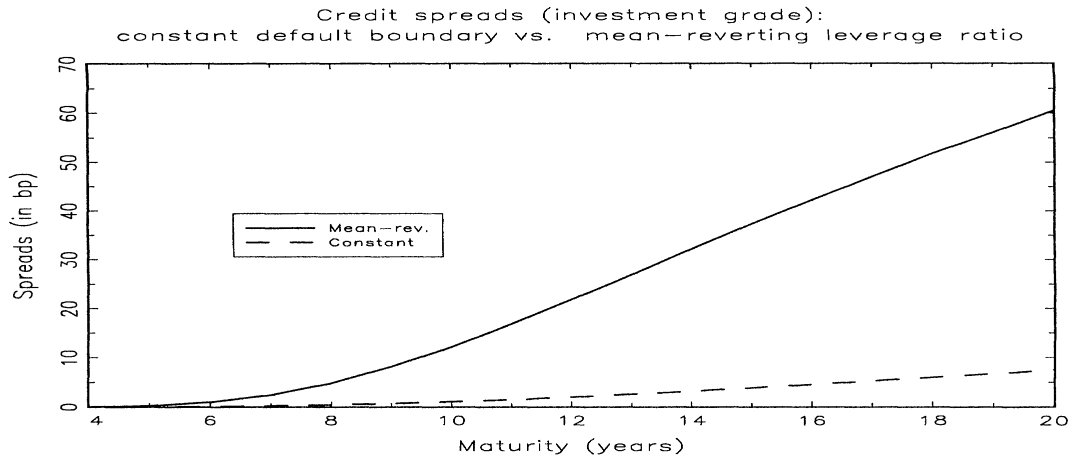


Figure 4. Investment-grade credit spreads for constant boundary versus mean-reverting leverage ratio. In both models we set $C = .075$, $r_0 = .06$, $\theta = .06$, $\kappa = .1$, $\eta = .015$, $\delta = .03$, $\sigma = .2$, $\rho = -.2$. For the stationary leverage model $\lambda = .18$, $\nu = 0.6$, $\phi = 2.8$. As before $\omega = .56$, $\omega_{coup} = 1$. The initial leverage is set to 15 percent.

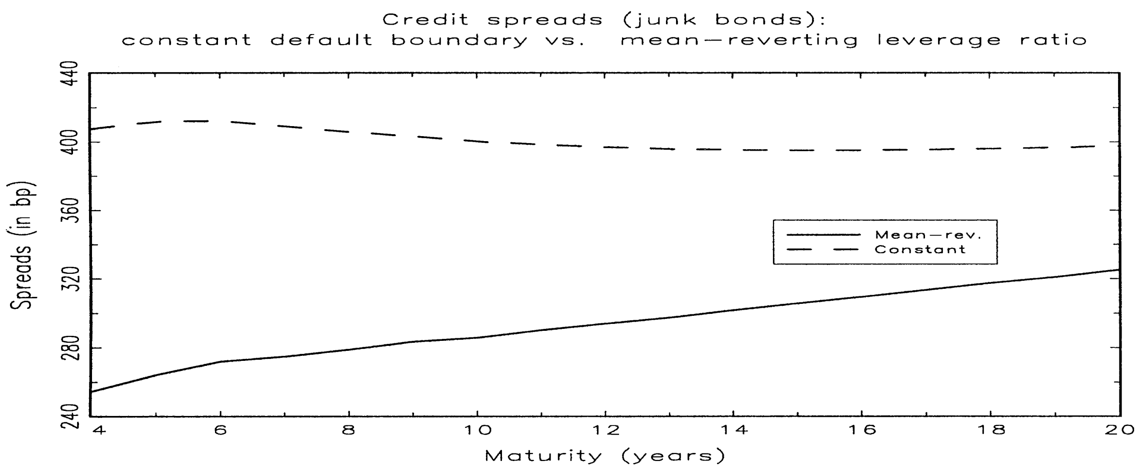


Figure 5. Speculative-grade credit spreads for constant default boundary versus mean-reverting leverage ratio case. In both models we set $C = .075$, $r_0 = .06$, $\theta = .06$, $\kappa = .1$, $\eta = .015$, $\delta = .03$, $\sigma = .2$, $\rho = -.2$. For the stationary leverage model $\lambda = .18$, $\nu = .5$, $\phi = 2.8$. As before $\omega = .56$, $\omega_{coup} = 1$. The initial leverage is set to 65 percent.

Source: Collin-Dufresne and Golstein 2001

Countercyclical Risk Premium

- Huang and Huang (2003) also extends previous model to consider time varying risk premia.

$$\begin{aligned} dV_t &= (\pi_t + r_t - \delta_t)V_t dt + \sigma V_t dW_1 && (1)\\ dr_t &= (\xi - \beta r_t)dt + \nu dW_2 && (2)\\ d\pi_t &= = k_\pi(\overline{\pi} - \pi_t)dt + \sigma_\pi dW_3 && (3) \end{aligned}$$

- Again, state variable $lnX_t = ln(V_t) - ln(V_t^*)$ would follow a joint gaussian process together with r_t and π_t.
  - Proof of original Longstaf and Schwartz can then be generalized to this case. It is just more complicated
- $\Longrightarrow$ bond pricing formula still the same, with $Q(X, r, \pi, t; T)$ that depends also on π

Table 7: A Counter-Cyclical Market Risk Premium

Panel A: Maturity = 10 years

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	Target			*Implied*				
Credit Rating	Leverage Ratio (%)	Equity Prem (%)	Cum. Default Prob. (%)	Asset Vol. (%)	Asset Risk Prem (%)	**Calc Credit Spread** (bps)	Avg Yield Spread (bps)	% of Spread due to Default
Aaa	13.1	5.37	0.77	33.2	4.96	13.1	63	20.8
Aa	21.2	5.60	0.99	29.4	4.92	18.2	91	20.0
A	32.0	5.99	1.55	26.5	4.90	28.8	123	23.4
Baa	43.3	6.55	4.39	26.7	5.04	65.2	194	33.6
Ba	53.5	7.30	20.63	33.0	5.53	202.7	320	63.3
B	65.7	8.76	43.91	39.9	6.48	392.9	470	83.6

Panel B: Maturity = 4 years

Aaa	13.1	5.38	0.04	37.1	4.95	1.6	55	3.0
Aa	21.2	5.60	0.23	35.3	4.90	7.9	65	12.1
A	32.0	5.99	0.35	30.7	4.86	12.8	96	13.4
Baa	43.3	6.55	1.24	29.7	4.93	38.7	158	24.5
Ba	53.5	7.30	8.51	35.0	5.34	186.1	320	58.1
B	65.7	8.76	23.32	40.0	6.30	458.7	470	97.6

Table 7 reports calibration results and calculated credit yield spreads for two maturities under the assumption that market asset risk premia are counter-cyclically time varying. The modeling and parameter setup is otherwise identical to that of the base case. For bonds of each credit rating and maturity, the model calibrated to match the target initial leverage ratio (column 2), equity premium (column 3), and cumulative default probability until bond maturity (column 4) is shown to have the implied asset volatility and the asset risk premium given in columns 5 and 6. The credit yield spreads calculated from such calibrated models (column 7) are also shown as a percentage (column 9) of historically observed yield spreads for the bonds (column 8). Parameter choices are as follows: the coupon rate is 8.13%; the asset payout ratio $\delta = 6\%$; the interest rate is constant at 8%; the default boundary V^* equals 60% of the total face value of the firms' bonds outstanding; the recovery rate given default is fixed at 51.31%; and the parameters in the risk-premium process are $\kappa_\pi = 0.202$, $\sigma_\pi = 3.1\%$, and $\rho_{\pi V} = -0.35$.

Source: Huang Huang 2003

Models with Endogenous Default

- So far the threshold V^* has been assumed exogenously.
- Leland (1994), Leland and Toft (1996) and others have assumed that V^* is chosen optimally to maximize firm value.
- Consider any claim on the firm that pays a non-negative total coupon rate C.
- If $F(V,t)$ denotes the value of this claim at time t, same argument as above imply

$$\frac{1}{2}\sigma^2 V^2 F_{VV}(V,t) + rVF_V(V,t) - rF(V,t) + F_t(V,t) + C = 0 \tag{4}$$

- Consider a security without any time-dependency (e.g. consol bonds) so that $F_t(t,V) = 0$.
- Then $F(V)$ has to satisfy the ODE

$$\frac{1}{2}\sigma^2 V^2 F_{VV}(V) + rVF_V(V) - rF(V) + C = 0 \tag{5}$$

- The general solution is known to be

$$F(V) = A_0 + A_1 V + A_2 V^{-\gamma} \tag{6}$$

- where $\gamma = 2r/\sigma^2$.
- The constants A_0, A_1 and A_2 are determined by boundary conditions.

Endogenous Default Models: Debt

- Firms have in incentive to issue debt to take advantage of the tax shields on interest expenses.
- Denote by V_B the default threshold, and assume debt holders get $(1-\alpha)V_B$ if bankruptcy occurs.
- Let $d(V,C)$ be the value of debt for $V \geq V_B$.
- It must solve the ODE (5) and hence has general solution (6).
- The boundary conditions are

$$\begin{aligned} d(V,C) &= (1-\alpha)V_B \text{ at } V = V_B \\ d(V,C) &\to C/r \text{ as } V \to \infty \end{aligned}$$

- Hence, from (6) we immediately find that

$$A_1 = 0;\ \ A_0 = \frac{C}{r}\ \ A_2 = \left((1-\alpha)V_B - \frac{C}{r}\right)V_B^{\gamma}$$

- Hence

$$d(V,C) = \frac{C}{r}\left(1 - \left(\frac{V}{V_B}\right)^{-\gamma}\right) + (1-\alpha)V_B\left(\frac{V}{V_B}\right)^{-\gamma}$$

- The term $\left(\frac{V}{V_B}\right)^{-\gamma}$ has the interpretation of being the NPV of \$1 at that stochastic time τ when bankruptcy occurs.
- Hence, the value of risky debt equals a weighted average of the value of a risk-free consol bond (C/r) and the fraction of recovery bond holders can obtain due to bankruptcy.

Endogenous Default Models: Firm Value

- Given the asset values V, we can compute the total value of the firm.
- Debt issuance affect this value in two ways
  1. It reduces the value due to bankruptcy costs;
  2. It increases firm value due to the tax deductibility of interest payments
- Value of the firm is then

$$v(V) = V + TB(V) - BC(V)$$

- Consider a security that pays no coupon but value at V_B equal to the bankruptcy costs αV_B.
- As this security satisfies the ODE (5) with $C = 0$.
- Let $BC(V)$ denote the value today of this security. The boundary conditions are

$$\begin{aligned} BC(V) &= \alpha V_B \text{ at } V = V_B \\ BC(V) &\to 0 \text{ as } V \to \infty \end{aligned}$$

- Hence, in this case we obtain

$$BC(V) = \alpha V_B \left(\frac{V}{V_B} \right)^{-\gamma}$$

- $BC(V)$ is then a decreasing, strictly convex function of V.

Endogenous Default Models: Firm Value

- Finally, consider now the value of the tax benefits associated with debt financing.
- Let θ be the tax rate.
- The tax benefits resemble a security that pays a constant coupon equal to θC (as long as $V > V_B$).
- Denote this security by $TB(V)$. It must satisfy (5) with boundary conditions

$$\begin{aligned} TB(V,C) &= 0 \text{ at } V = V_B \\ TB(V,C) &\to \frac{\theta C}{r} \text{ as } V \to \infty \end{aligned}$$

- We then now obtain

$$TB(V,C) = \frac{\theta C}{r}\left(1 - \left(\frac{V}{V_B}\right)^{-\gamma}\right)$$

- Tax benefits are an increasing and strictly concave function of V.
- Notice that here we are assuming that a firm always benefits fully (i.e. in amount θC) from tax deductibility of coupon payments.
- In fact, this is true only if $EBIT > C$. This case can be also modeled for the case where $EBIT = \lambda V$ (that is, earnings are proportional to asset value), but we do not do it.

Endogenous Default Models: Firm and Equity Value

- The total value of the firm is then given by

$$\begin{aligned} v(V,C) &= V + TB(V,C) - BC(V) \\ &= V + \frac{\theta C}{r}\left(1 - \left(\frac{V}{V_B}\right)^{-\gamma}\right) - \alpha V_B \left(\frac{V}{V_B}\right)^{-\gamma} \end{aligned}$$

- If $C > 0$ and either $\alpha > 0$ or $\theta > 0$, then v(V, C) is strictly concave in V.
- If both $\alpha > 0$ and $\theta > 0$, we also have that

$$\begin{aligned} v(V) &< V \text{ as } V \to V_B \\ v(V) &> V \text{ as } V \to \infty \end{aligned}$$

- Hence, volatility of firm value is higher for low V and lower for high V than σ.
- The value of equity is finally

$$\begin{aligned} e(V,C) &= v(V,C) - d(V,C) \\ &= V + (\theta - 1)\frac{C}{r} + \left[(1-\theta)\frac{C}{r} - V_B\right]\left(\frac{V}{V_B}\right)^{-\gamma} \end{aligned}$$

- When V_B is optimally determined in the interest of equity holders, it turns out that $(1-\theta)\frac{C}{r} - V_B > 0$.
- Hence, $e(V, C)$ is convex in V.

Optimal Default

- If there are no covenants, bankruptcy will occur when it is in the best interest of equity holders to go bankrupt (given the value of assets V).
- That is, bankruptcy will occur only when the firm cannot meet the required coupon payments by issuing additional equity. That is, when equity is zero.
- However, clearly, for any value V_B that triggers bankruptcy we have zero equity value.
- Notice that the total firm value $v(V)$ is maximized by choosing V_B as low as possible.
- However, not all values are feasible, because we must also ensure that $e(V,C) > 0$ for all V (limited liability).
- Since $e(V,C)$ is strictly convex if $V_B < (1-\theta)\, C/r$, the lowest possible value for V_B that can be set such that $e(V,C) > 0$ for all $V > V_B$ is at the point where

$$\frac{de(V,C)}{dV}|_{V=V_B} = 0$$

- This is the (standard) smooth pasting condition.
- By solving this equation one finds

$$V_B = \frac{(1-\theta)\, C/r}{\gamma/(1+\gamma)} = \frac{(1-\theta)\, C}{r + \frac{1}{2}\sigma^2}$$

Table 4: Endogenous Default with the Leland-Toft Model

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	Target			*Implied*					
Credit Rating	Leverage Ratio (%)	Equity Prem (%)	Cum. Default Prob (%)	Recovery as % of V^*	Asset Vol. (%)	Asset Risk Prem (%)	**Calc Credit Spread** (bp)	Avg Yield Spread (bp)	% of Spread due to default
Aaa	13.1	5.37	0.77	100.00	34.06	4.74	36.89	63	58.6
Aa	21.2	5.60	0.99	96.79	29.23	4.51	34.46	91	37.9
A	32.0	5.99	1.55	87.72	25.25	4.17	38.50	123	31.3
Baa	43.3	6.55	4.39	87.29	25.05	3.92	59.46	194	30.6
Ba	53.5	7.30	20.63	100.00	36.00	4.68	165.70	320	51.8
B	65.7	8.76	43.91	100.00	52.33	6.68	408.38	470	86.9

Table 4 reports calibration results and calculated credit yield spreads using the Leland and Toft (1996) model. The setup is otherwise identical to that of the base case. For bonds of each credit rating, the model calibrated to match the target initial leverage ratio (column 2), equity premium (column 3), and cumulative default probability over 10 years (column 4) is shown to have the implied asset volatility and the asset risk premium given in columns 6 and 7. The bond recovery rate given default is assumed to be the lower of 51.31% of the face value or 100% of the firm value at default (V^*). Bond recovery as a percentage of the firm value at default, V^*, is shown in column 5. The credit yield spreads calculated from such calibrated models (column 8) are also shown as a percentage (column 10) of historically observed yield spreads for the bonds (column 9). Parameter choices are as follows: the coupon rate is 8.13%; the asset payout ratio $\delta = 6\%$; and the interest rate is constant at 8%.

Note that the higher credit yield spreads here are due to the fact that the version of LT model that we implement here assumes an infinite bond maturity, even though we calibrate the model to match only the 10-year cumulative default probability. In fact, the credit yield spreads for 10-year bonds generated by such a calibrated model are indeed very close to those shown in the base case. See section 4.3 for more details.

Source: Huang Huang 2003

Duffie and Lando (2001): Incomplete Accounting Information

- The Merton model above has a disturbing characteristics.
- As the maturity of debt goes to zero, credit spread must necessarily go to zero as well.
- The intuition is clear: If $V_t > V_B$, the probability that default will occur in the time interval $T - t$ decreases with $T - t$.
- In the limit, the bond becomes (locally) risk-less.
- Duffie and Lando (2001) introduce incomplete information in a model almost identical to the previous one, although with a few important caveat.

  1. First, they assume that the firm generates cash flows at the rate δV.
     - This case is also covered in Leland (1994), who shows that all the results go through.
  2. More substantially, given that DL assume incomplete information of agents on the asset value of the firm, the usual risk-neutral approach to solving for debt cannot be used. Hence, they simply assume that investors *are* risk-neutral.

Duffie and Lando (2001): Model

- So, let again

$$\frac{dV_t}{V_t} = \mu dt + \sigma dB_t$$

- At time $t = 0$ the debt is sold for D (to be determined).
- Suppose that the debt is issued at par, this implies that its face value is D and the coupon rate is C/D (as a percentage of par).
- Assume finally that the proceeds from the sale of debt are paid at time $t = 0$ as a cash distribution to initial equity holders.
- Given these assumptions, the net present value of cash flows generated by the assets (excluding the effects of liquidation losses and tax shields) is

$$E_t\left[\int_t^\infty e^{-r(s-t)} \delta V_s ds | V_t\right] = \frac{\delta V_t}{r-\mu} \tag{7}$$

- provided that $r > \mu$.

Duffie and Lando (2001): Optimal Liquidation

- As before, assume the firm is operated by managers in the full interest of equity owners.
- The only choice of equity owners is when to liquidate the firm.
- Hence, a liquidation policy is given by a $\{\mathcal{F}_t\}$-measurable stopping time $\tau : \Omega \rightarrow [0, \infty]$.
- Let V_τ be the asset level at liquidation.
- The *liquidation value* can then be defined as the unlevered value defined in (7)

$$\frac{\delta V_\tau}{r - \mu}$$

- This assumption is made for simplicity.
- Let α be the fraction of assets lost due to frictions, so that now debt holders are then assigned the remaining assets

$$(1 - \alpha) \frac{\delta V_\tau}{r - \mu}$$

- The value of equity to shareholders at time $t = 0$ is then given by

$$F(V_0, C, \tau) = E\left[\int_0^\tau e^{-rt} (\delta V_t + (\theta - 1) C)\, dt\right]$$

- Equity holders will then choose the policy τ such that

$$S_0 = \max_\tau F(V_0, C, \tau)$$

Duffie and Lando (2001): Solving for Optimal Liquidation

- Specifically, we can conjecture that the value of equity at any point in time

$$\begin{aligned} S_t &= \sup_{\tau} E\left[\int_t^{\tau} e^{-r(s-t)} \left(\delta V_s + (\theta - 1) C\right) dt | \mathcal{F}_t\right] \\ &= e(V_t) \end{aligned}$$

- Notice that holding this security implies

$$dS_t = e'(V_t)\,\mu V_t dt + \frac{1}{2} e''(V_t)\,\sigma^2 V_t^2 + e'(V_t)\,V_t \sigma dB_t$$

- Since agents are risk neutral, we must have that the expected instantaneous payoff from keeping the firm operating equal the instantaneous cost (=interest rate).
- That is

$$E\left[dS_t + \delta V_t + (\theta - 1) C\right] = S_t r$$

- which implies the ordinary differential equation

$$e'(V)\,\mu V + \frac{1}{2} e''(V)\,\sigma^2 V^2 - re(V) = (1 - \theta) C - \delta V \tag{8}$$

- whenever $V > V_B$.
- This is only slightly more complicated ordinary differential equation than the one obtained before and it can again be solved in closed form.

Duffie and Lando (2001): Solving for Optimal Liquidation

- As before, impose $e(V) = 0$ if $V \leq V_B$
- and the "smooth pasting" condition $e'(V_B) = 0$
- The solution to the ODE with these boundary conditions yields

$$V_B(C) = \frac{(1-\theta) C\gamma (r-\mu)}{r(1+\gamma)\delta}$$

- where, defining $m = \mu - \frac{1}{2}\sigma^2$, $\gamma = \frac{m+\sqrt{m^2+2r\sigma^2}}{\sigma^2}$.
- Finally

$$e(V,C) = \frac{\delta V}{r-\mu} - \frac{V_B(C)\delta}{r-\mu}\left(\frac{V}{V_B(C)}\right)^{-\gamma} + (\theta-1)\frac{C}{r}\left[1-\left(\frac{V}{V_B(C)}\right)^{-\gamma}\right]$$

- We can interpret this equation as before

1. The first term is the NPV of future cash flows generated by assets, assuming no liquidation;
2. The second term is NPV of the cash flows lost because of liquidation;
3. The last term is the NPV of the cost of future debt coupon payments, net of tax shields (recall that $\theta < 1$).

Duffie and Lando (2001): Debt

- From the above, it is also immediate to find that the expected present value of cash flows to the bond at any time t (before liquidation) is given by

$$d(V,C) = \frac{(1-\alpha)\,V_B(C)\,\delta}{r-\mu}\left(\frac{V}{V_B(C)}\right)^{-\gamma} + \frac{C}{r}\left[1-\left(\frac{V}{V_B(C)}\right)^{-\gamma}\right]$$

- At time $t=0$ the choice of the coupon rate C is going to be made to maximize the market value of equity and debt, given asset values V_0.
- That is, C is chosen to solve

$$\max_C \{e(V_0,C) + d(V_0.C)\}$$

- We finally notice that the results obtained in the previous section are a special case of these ones where we set $\mu = r - \delta$, and $\delta = 0$.

Duffie and Lando (2001): Imperfect Information

- Bond holders cannot observe directly V_t while managers (equity holders) have perfect information.
- They then know that managers will default as soon as V_t reaches V_B, a value that bond holders can indeed compute.
- However, they cannot assess the distance of V_t from V_B.
- One first approach could be to apply our learning results of the previous TNs.
- Suppose that bond holders observe a continuous time signal

$$ds_t = V_t dt + \sigma_s dB_t^2$$

- If one assumes that at time $t = 0$ V_0 was normally distributed with mean $\overline{V}_0$ and variance $\sigma_{V,0}^2$ one then has that V_t is also normally distributed, with mean $\overline{V}_t$ and some variance $\sigma_{V,0}^2$, where

$$d\overline{V}_t = \mu \overline{V}_t dt + \overline{\sigma}_{V,t} dB_t$$

- However, this is not correct. In fact, at any point in time t we have to condition also on the event that *default has not occurred yet.*
- This implies that the left tail of the posterior distribution on V_t cannot be below V_B.
- This complication makes the “learning” challenging.

Duffie and Lando (2001): Learning

- Assume that investors can observe a noisy signal of V_t only every so often, say quarterly.

$$Y_t = \log\left(\widehat{V}_t\right) = \log\left(V_t\right) + U_t$$

- where U_t (the noise) is normally distributed and independent of $Z_t = \log\left(V_t\right)$.
- Hence, the information available in the secondary market at time t is

$$\mathcal{H}_t = \sigma\left(\left\{Y_{t_1}, ..., Y_{t_n}, 1_{\{\tau \leq s\}} : 0 \leq s \leq t\right\}\right)$$

- for the largest n such that $t_n \leq t$, where $\tau = \tau\left(V_B\right)$.
- **Assumptions:** (1) Equity is not traded on the public market; (2) Equity owners are precluded from trading in public debt markets.
- They use the following results. Define

$$\psi\left(z, x, \sigma\sqrt{t}\right) = \Pr\left(\min\left(Z_s : 0 \leq s \leq t\right) > 0 | Z_0 = z \;\&\; Z_t = x\right)$$

- that is, the probability that if Z_s starts from z at $s = 0$ and reaches x at time $s = t$, it has been always positive.
- It turns out that this probability is given by

$$\psi\left(z, x, \sigma\sqrt{t}\right) = 1 - \exp\left(-\frac{2zx}{\sigma^2 t}\right)$$

Duffie and Lando (2001): Learning

- Then, conditional on the observation of $Y_t = Z_t + U_t$ they compute the density $b(.|Y_t, z_0, t)$ of Z_t, killed at $\tau = \inf\{t : Z_t \leq \underline{v}\}$, (where $\underline{v}$ is the log optimal default boundary) that is

$$b(x|Y_t, z_0, t)\,dx = \Pr(\tau > t \,\&\, Z_t \in dx|Y_t) \text{ for } x > \underline{v} :$$

- They show that Bayes rule implies

$$b(x|Y_t, z_0, t) \;=\; \frac{\phi_U(Y_t - x)\,\phi_Z(x)}{\phi_Y(Y_t)}\psi\left(z_0 - \underline{v}, x - \underline{v}, \sigma\sqrt{t}\right)$$

- where ϕ_U ϕ_Z and ϕ_Y are the densities of U_t, Z_t and Y_t respectively.
- The mean and variances of all these densities are all known (see Duffie and Lando (2001).
- Hence, one can finally compute

$$P(\tau > t|Y_t) = \int_{\underline{v}}^{\infty} b(z|Y_t, z_0, t)\,dz$$

- Finally, the density of Z_t conditional on Y_t and $\tau > t$ is

$$g(x|Y_t, z_0, t) = \frac{b(x|Y_t, z_0, t)}{\int_{\underline{v}}^{\infty} b(z|Y_t, z_0, t)\,dz}$$

- It turns out that one can actually compute this density in closed form.
- The formula is very messy and I do not want to report it here.

Duffie and Lando (2001): Default Probability

- Let the survival probability

$$\begin{aligned} p(t,s) &= \Pr(\tau > s|\mathcal{H}_t) \\ &= \int_{\underline{v}}^{\infty} (1-\pi(s-t,x-\underline{v}))\, g(x|Y_t,z_0,t)\, dx \end{aligned}$$

- where $\pi(t,x)$ denotes the probability of first passage of a Brownian motion with drift $m = \mu - \sigma^2/2$ and volatility parameter σ from an initial condition $x > 0$ to a level below 0 before time t.
- This density is also known explicitly.
- Hence, $1 - p(t,s)$ is the default probability between t and $s > t$.
- One can then define the default intensity as

$$\lambda_t = \lim_{h \downarrow 0} \frac{1 - p(t,t+h)}{h}$$

- Duffie and Lando show

$$\lambda_t(\omega) = \frac{1}{2}\sigma^2 \frac{\partial f(t,x,\omega)}{\partial x}|_{x=\underline{v}}$$

- if $t < \tau$, where $f(t,x,\omega)$ is the density of Z_t conditional on the information $\mathcal{H}_t$.
- This result shows that intensity based models can be obtained from structural models with incomplete accounting information.

Duffie and Lando (2001): Credit Spreads across maturity

- In the set up assumed above the firm is only issuing long-term debt (in fact, a consol).
- Hence, there is little room to talk about the maturity of bonds.
- However, in the secondary market we can think of a (default free) investment bank that strips the consol coupon bond of the firms to sell individual securities with all the possible maturities.
- In this case, it is still legitimate to compute the price of short term zero coupon bonds.
- Consider then a zero coupon bond with maturity s.
- Let $R(s)$ be the recovery rate of the principal of the bond if default occurs.
- The price at time t of such a bond is then given by

$$\varphi(t,s) = e^{-r(s-t)}p(t,s) - R(s)\int_t^s e^{-r(u-t)}p(t,du)$$

- The first term is the NPV of payoff contingent on no-default before s, while the second is the NPV of the recovery rate in case of default.
- Figure 3 from Duffie and Lando (2001) plot three different densities for $f_t(V_t)$ corresponding to different levels of expected asset value last year .

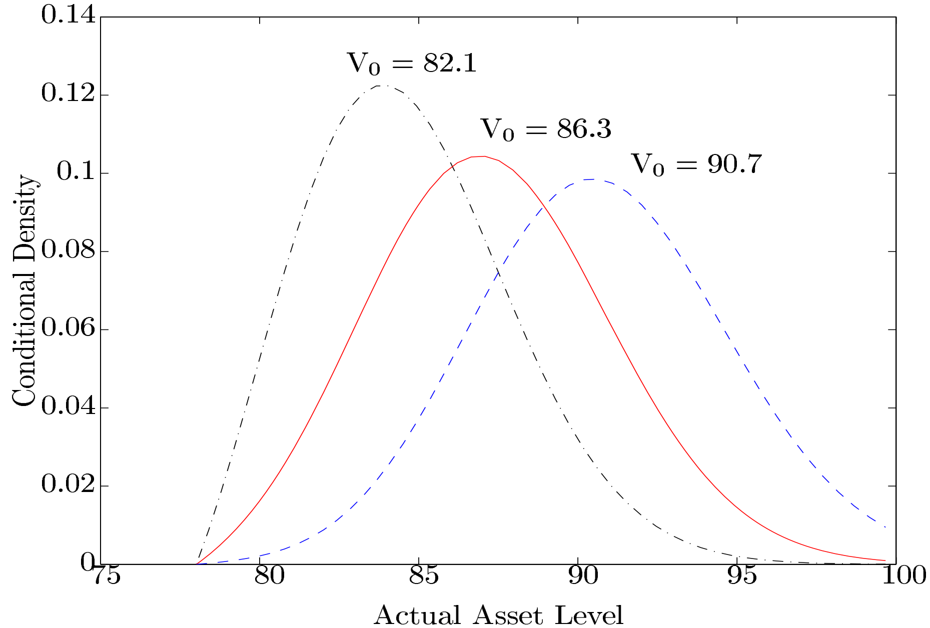


Figure 3: Conditional asset density, varying previous year asset level.

Source: Duffie and Lando (Econometrica, 2001)

- The idea is that for $V_0 = 90.7$, the slope of $f(V_t)$ is very small at $V_b = 78$ and therefore there is a small probability of default occurring in the next instant dt.
- Instead, for $V_0 = 82.1$, the slope of $f(V_t)$ at $V_b = 78$ is very high, which implies that agents would asses a relatively high probability of default in the next instant dt.
- Figure 7 in Duffie and Lando reports the implied default intensity (probability of default per unit of time) implied by given asset value.

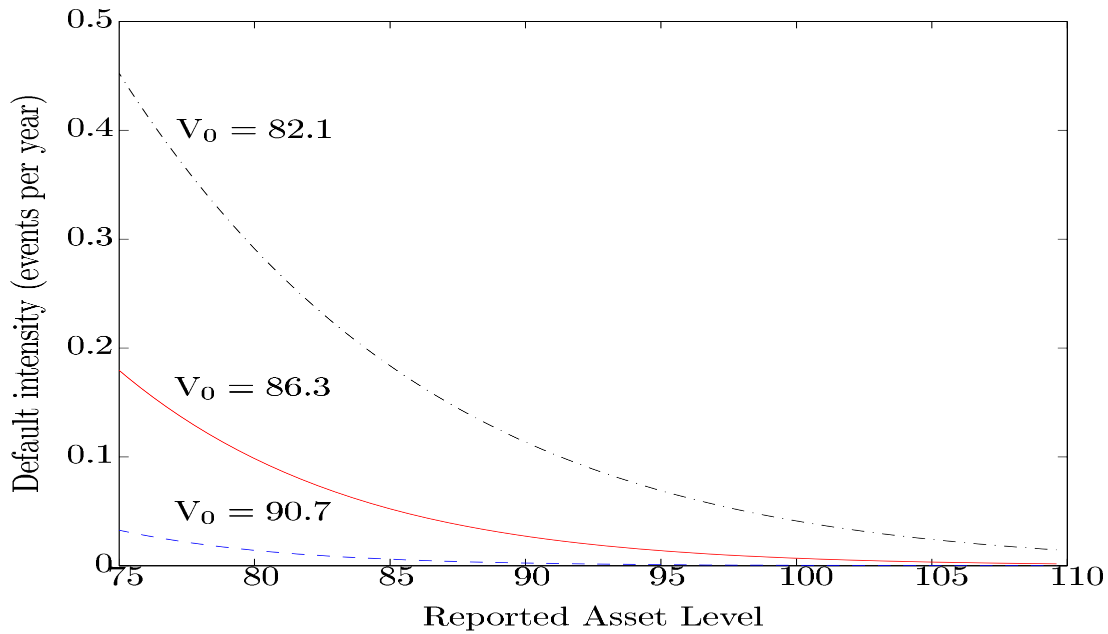


Figure 7: Default intensity, varying previous year asset level.

Source: Duffie and Lando (Econometrica, 2001)

- Figure 9 shows the implied credit spreads

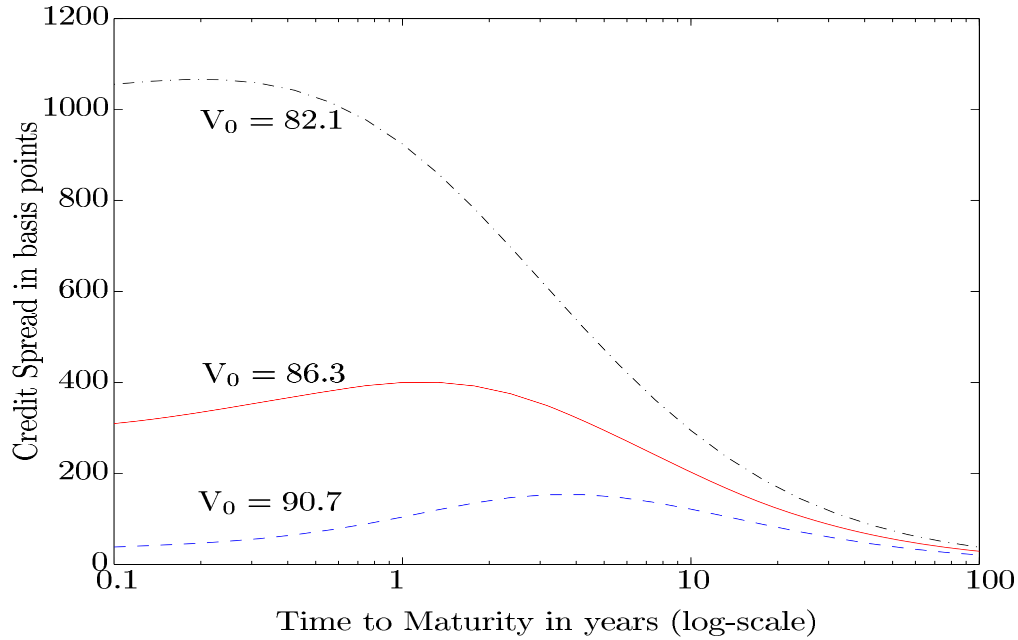
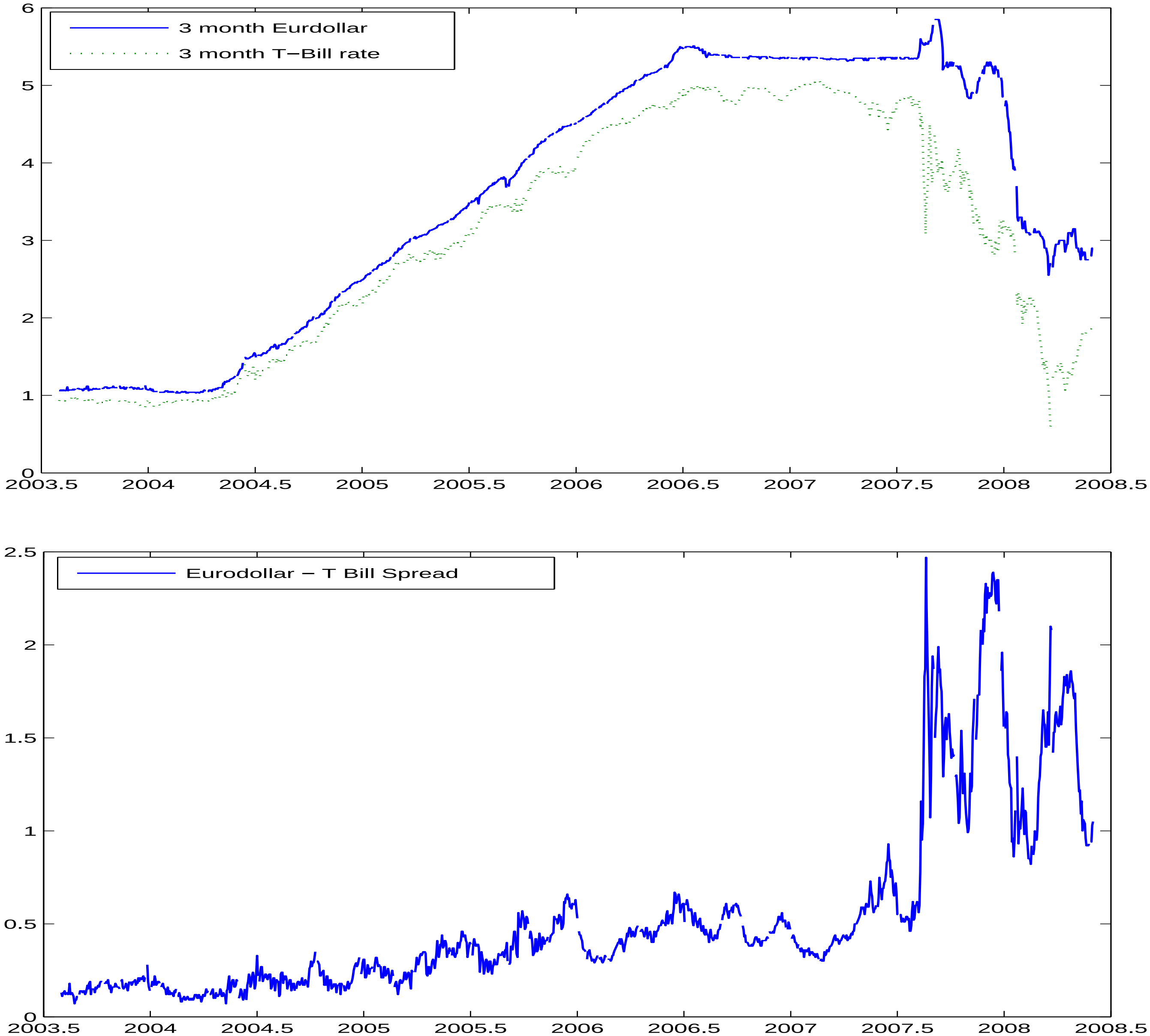


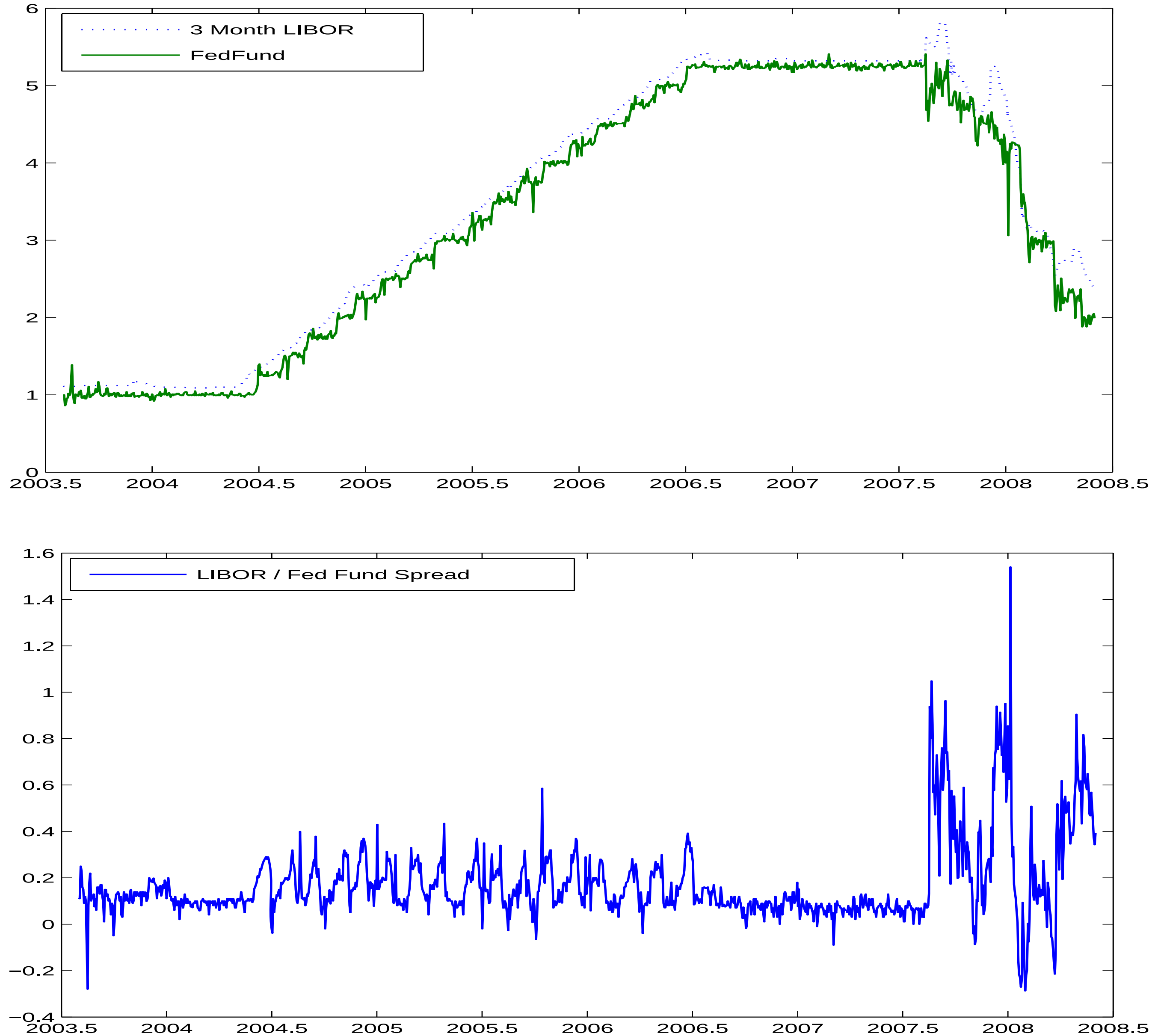
Figure 9: Credit spreads, varying previous year's asset level.

Source: Duffie and Lando (Econometrica, 2001)

Eurodollar vs. Treasury rates



LIBOR versus Fed Fund



Time Varying Credit Spreads

- Two recent papers tackle the issue of time varying credit spreads
  1. Chen, Collin-Dufresne and Goldstein (RFS, 2008) $\Longrightarrow$ Habit Formation;
  2. Hui Chen (2008) $\Longrightarrow$ EZ Preferences
  - Note that credit spreads can increase because of three reasons:
    1. probability of default increases
    2. (expected) recovery rate decreases
    3. market price of risk increases
  - The time varying market price of risk took care of the last of these quantities.

CCDG: Habit Formation

- Simplest version of Merton model
  - Exogenous time varying firm value $dV = V(\mu - \delta)dt + V\sigma dW$
  - Zero coupon bond debt
  - Default only at maturity T if $V < B$
- Use Campbell and Cochrane setting
  - marginal utility $m_t = e^{-\rho t} C_t^{-\gamma} S_t^{-\gamma}$
  - log surplus $s_t = log(S_t)$ follows mean reverting process

$$\Delta s_t = k(\overline{s} - s_t)\Delta t + \sigma\lambda(s)\Delta z$$

where, defining $s_{max} = \overline{s} + 0.5(1 - \overline{S}^2)$, we have

$$\lambda(s) = \frac{1}{\overline{S}}\left[\sqrt{1 - 2(s - \overline{s})} - 1\right] \quad \text{for} \quad s > s_{max}$$

  - Dividend and consumption follow the processes

$$\begin{aligned} \Delta c &= g_c \Delta t + \sigma_c \Delta z \\ \Delta d &= g_d \Delta t + \sigma_d \left(\rho_{cd}\Delta z + \sqrt{1 - \rho_{cd}^2}\Delta z_d\right) \end{aligned}$$

CCDG: Habit Formation

- A few issues: First, Campbell and Cochrane obtain the claim to aggregate dividends, instead of a claim to aggregate dividends and interest (equal to value of firm).
  - CCDG introduce a new aggregate "output process" ε_t (same form as dividends) to obtain a claim to total output.
- Second: the methodolgy yields the value of the aggregate market, and not of the individual firm.
  - CCDG price a beta 1 security, whose value P_t is obtained from

$$\frac{\Delta P_t}{P_t} = \frac{\Delta V_t}{V_t} + \sigma_{Idio} \Delta z_{Idio}$$

- Calibration is made to match:
  - Aggregate data: the usual suspects (as in CC)
  - Individual firms: Expected default frequency, Average Recovery Rate, Average Sharpe Ratio

		Baa			Aaa			
$s(0)$	Population Distr. (%)	Spread over Treas. (bp)	Q-Default Rate (%)	P-Default Rate (%)	Spread over Treas. (bp)	Q-Default Rate (%)	P-Default Rate (%)	Baa-Aaa Spread (bp)
-2.96	0.06	91.3	6.61	1.22	6.0	0.40	0.03	85.3
-2.86	0.07	90.1	6.50	1.36	5.9	0.39	0.04	84.2
-2.76	0.09	88.0	6.30	1.43	5.8	0.37	0.04	82.2
-2.66	0.11	85.5	6.12	1.54	5.3	0.34	0.04	80.2
-2.56	0.13	80.5	5.77	1.75	5.0	0.32	0.05	75.5
-2.46	0.15	76.0	5.46	1.89	4.8	0.31	0.05	71.2
-2.36	0.14	66.2	4.76	2.08	4.0	0.25	0.06	62.2
-2.27	0.04	52.9	3.82	2.20	3.0	0.17	0.06	49.9
Average		82.3	5.90	1.55	5.2	0.34	0.04	77.1
Std. Dev.		12.7	0.90	0.41	1.0	0.07	0.01	11.7

Table 4: Model generated four-year Baa and Aaa credit spreads as a function of initial log-surplus consumption ratio $s(0)$ for the constant default boundary case. Without loss of generality, initial firm value is normalized to one. Default boundary is specified to be independent of $s(0)$, and constant over time ($B_{Baa} = 0.356$, $B_{Aaa} = 0.208$). Population averages over the steady state distribution are then determined.

		Baa			Aaa			
$s(0)$	Population Distr. (%)	Spread over Treas. (bp)	Q-Default Rate (%)	P-Default Rate (%)	Spread over Treas. (bp)	Q-Default Rate (%)	P-Default Rate (%)	Baa-Aaa Spread (bp)
-2.96	0.06	155.0	27.05	3.74	47.3	9.15	0.42	107.7
-2.86	0.07	149.9	26.22	3.86	44.5	8.58	0.47	105.4
-2.76	0.09	144.7	25.33	4.40	42.4	8.17	0.54	102.3
-2.66	0.11	138.5	24.33	4.72	39.4	7.60	0.59	99.1
-2.56	0.13	131.8	23.21	5.20	36.6	7.05	0.66	95.2
-2.46	0.15	124.9	22.08	5.86	34.2	6.59	0.76	90.7
-2.36	0.14	116.4	20.71	6.64	30.8	5.93	0.91	85.6
-2.27	0.04	104.7	18.86	7.36	26.0	5.03	1.05	78.7
Average		141.8	24.88	4.89	42.5	8.19	0.63	99.4
Std. Dev.		26.8	4.44	1.39	13.9	2.69	0.22	13.5

Table 5: Model generated 10-year Baa and Aaa credit spreads as a function of initial log-surplus consumption ratio $s(0)$ for the constant default boundary case. Without loss of generality, initial firm value is normalized to one. Default boundary is specified to be independent of $s(0)$, and constant over time ($B_{Baa} = 0.295$, $B_{Aaa} = 0.171$). Population averages over the steady state distribution are then determined.

CCDG: Countercyclical Default Boundaries

- Results somewhat disappointing.
  1. AAA-Treasury way too low (liquidity premium? CCDG gave up from the start on this dimension)
  2. Baa-AAA better than in Merton model, but still too low.
- Intuition: Bond price is

$$P(T) = E\left[\frac{m_T}{m_0} X(T)\right] = E\left[\frac{m_T}{m_0}\right] E\left[X(T)\right] + cov\left(\frac{m_T}{m_0}, X(T)\right)$$

  – First term is determined by risk free rate and expected default frequency.
  – Credit spread depends on the covariance term: Need high marginal utility when cash flow is low.
  – Problem: Current specification does not consider the state-dependent nature of cash flows $X(T)$, which are low during bad states
- Solution: assume (exogenously) default boundaries gets higher when surplus is low.
  – Assume

$$B(S(0), S(t)) = \xi[0.52 - 0.61S(0)][(1 - slope(S(t) - \overline{S})]$$

  – Choose parameters to match default frequency, predictability of credit spreads onto default rates and stationary credit spreads.

		Baa			**Aaa**			
$s(0)$	Population Distr. (%)	Spread over Treas. (bp)	Q-Default Rate (%)	P-Default Rate (%)	Spread over Treas. (bp)	Q-Default Rate (%)	P-Default Rate (%)	Baa-Aaa Spread (bp)
-2.96	0.05657	151.1	10.65	1.74	11.4	0.80	0.06	139.7
-2.86	0.07159	140.4	9.92	1.72	10.4	0.73	0.05	130.0
-2.76	0.08983	127.6	9.05	1.65	9.4	0.65	0.05	118.2
-2.66	0.10887	112.4	8.00	1.57	7.9	0.54	0.05	104.5
-2.56	0.12837	94.6	6.77	1.48	6.8	0.46	0.04	87.8
-2.46	0.14735	78.2	5.62	1.34	5.5	0.36	0.02	72.7
-2.36	0.14429	56.1	4.05	1.21	3.9	0.24	0.04	52.2
-2.27	0.03753	33.6	2.43	0.98	2.1	0.11	0.02	31.5
Average		115.6	8.18	1.55	8.5	0.59	0.04	107.1
Std. Dev.		50.3	3.45	0.29	4.0	0.30	0.01	46.3

Table 7: Model generated four-year Baa and Aaa credit spreads as a function of initial log-surplus consumption ratio $s(0)$ for the counter-cyclical default boundary case. Without loss of generality, initial firm value is normalized to one. Default boundary is specified to be a function of both initial state of the economy and current state of the economy: $B_{Baa}(S(t), S(0)) = \psi^*_{Baa}[0.52 - 0.61S(0)][1 - slope * (S(t) - \overline{S})]$. The parameters $\{\psi_{Baa}, \psi_{Aaa}\}$ are chosen to match historical default rates. *Slope* is chosen to closely capture both the historical variation in spreads and the regression coefficient of spreads on future default rates. Population averages over the steady state distribution are then determined.

		Baa			**Aaa**			
$s(0)$	Population Distr. (%)	Spread over Treas. (bp)	Q-Default Rate (%)	P-Default Rate (%)	Spread over Treas. (bp)	Q-Default Rate (%)	P-Default Rate (%)	Baa-Aaa Spread (bp)
-2.96	0.06	211.4	35.26	4.55	68.6	13.01	0.55	142.8
-2.86	0.07	201.4	33.76	4.59	64.0	12.14	0.57	137.4
-2.76	0.09	189.0	31.98	4.73	59.2	11.25	0.59	129.8
-2.66	0.11	177.7	30.31	4.95	53.9	10.28	0.61	123.8
-2.56	0.13	163.0	28.09	5.12	48.7	9.31	0.65	114.3
-2.46	0.15	147.0	25.69	5.33	42.9	8.24	0.69	104.1
-2.36	0.14	130.3	23.13	5.38	36.4	7.02	0.73	93.9
-2.27	0.04	109.2	19.83	5.67	28.7	5.58	0.77	80.5
Average		181.8	30.79	4.87	58.0	11.03	0.61	123.8
Std. Dev.		48.1	7.13	0.56	22.4	4.13	0.10	26.3

Table 8: Model generated 10-year Baa and Aaa credit spreads as a function of initial log-surplus consumption ratio $s(0)$ for the counter-cyclical default boundary case. Without loss of generality, initial firm value is normalized to one. Default boundary is specified to be a function of both initial state of the economy and current state of the economy: $B_{Baa}(S(t), S(0)) = \psi^*_{Baa}[0.52 - 0.61S(0)][1 - slope * (S(t) - \overline{S})]$. The parameters $\{\psi_{Baa}, \psi_{Aaa}\}$ are chosen to match historical default rates. *Slope* is chosen to closely capture both the historical variation in spreads and the regression coefficient of spreads on future default rates. Population averages over the steady state distribution are then determined.

- What about the dynamics of credit spreads?
  - CCDG feed the model actual consumption data, and compute the variation in credit spreads implied by consumption.

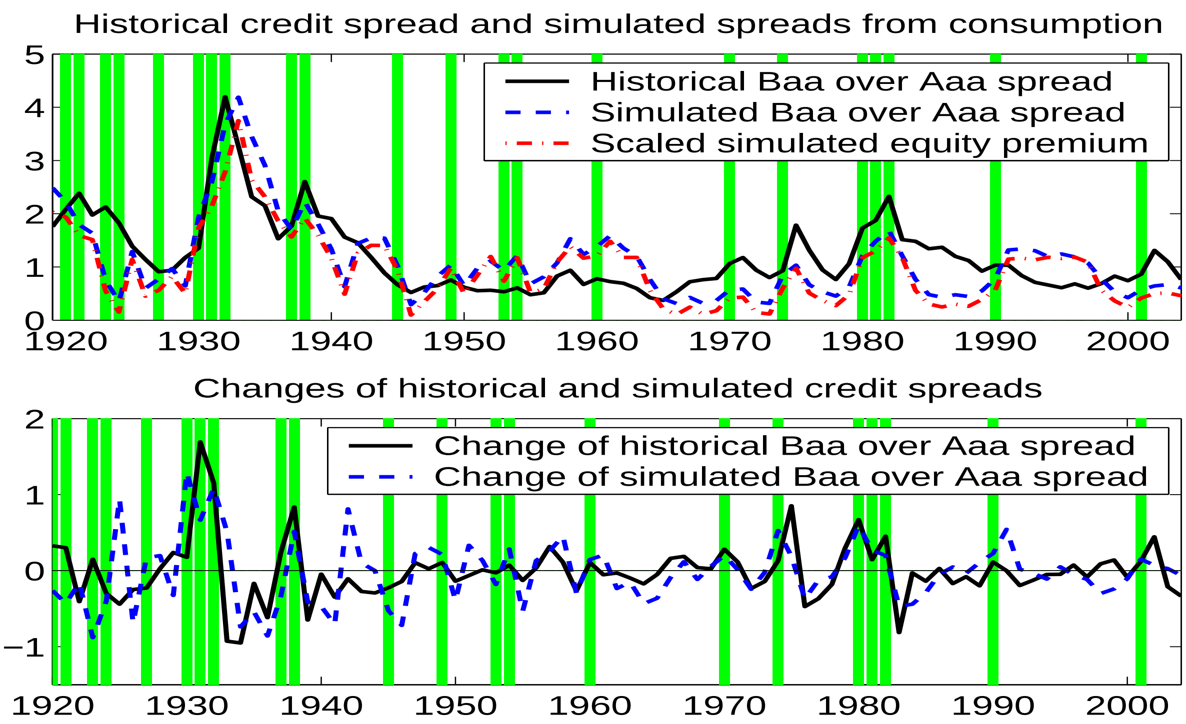


Figure 3: The levels and changes of historical and simulated credit spreads and simulated equity premium implied by the Campbell-Cochrane model. The simulated data is backed out from historical surplus consumption ratio. The scaled simulated equity premium is the simulated equity premium divided by eight.

Chen (2008): Endogenous Countercyclical Defaults

- Default Rates, recovery rates and credit spreads are countercyclical

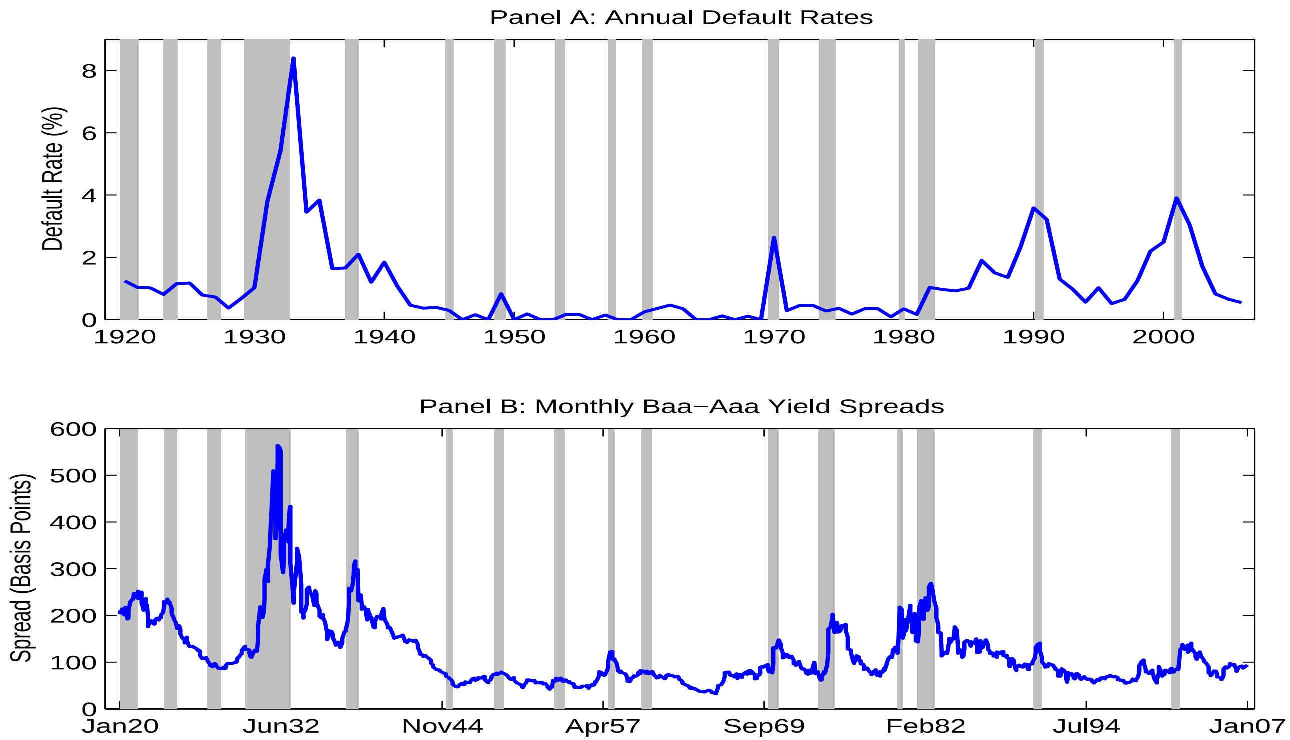


Figure 1: Annual Global Corporate Default Rates and Monthly Baa-Aaa Credit Spreads, 1920-2006. Shaded areas are NBER-dated recessions. For annual data, any calendar year with at least 5 months being in a recession as defined by NBER is treated as a recession year. Data source: Moody's.

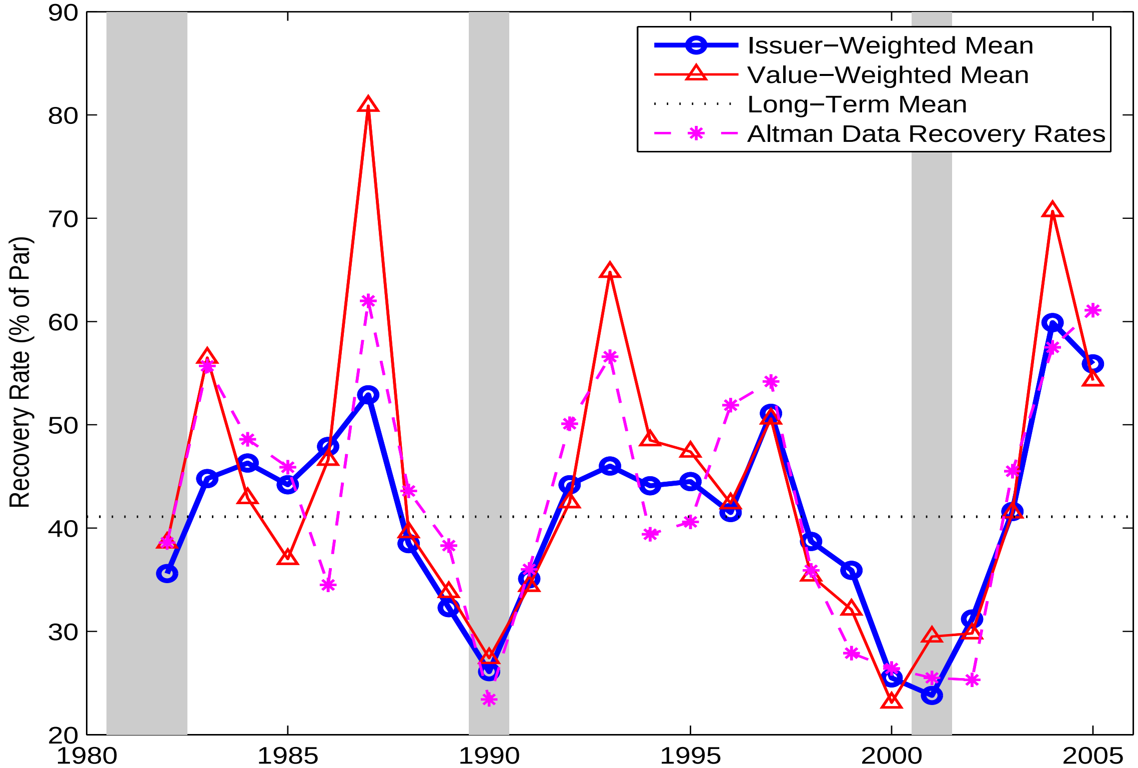


Figure 2: Annual Average Recovery Rates, 1982-2005. Issuer-weighted and value-weighted mean recovery rates are from Moody's. "Altman Data Recovery Rates" are from Altman and Pasternack (2006). Shaded areas are NBER-dated recessions.

Chen (2008): Economy

- time is continuous
- representative agent with recursive utility:

$$U_t = E_t \left(\int_t^{\infty} f\left(c_s, U_s\right) ds \right)$$

where

$$f\left(c, v\right) = \frac{\rho}{1 - \frac{1}{\psi}} \frac{c^{1-\frac{1}{\psi}} - \left(\left(1-\gamma\right) v\right)^{\frac{1-1/\psi}{1-\gamma}}}{\left(\left(1-\gamma\right) v\right)^{\frac{1-1/\psi}{1-\gamma}-1}}$$

  - ρ rate of time preference
  - γ coefficient of relative risk aversion
  - ψ intertemporal elasticity of substitution

- aggregate consumption Y_t:

$$\frac{dY_t}{Y_t} = \theta_m\left(s_t\right) dt + \sigma_m\left(s_t\right) dW_t^m$$

- state variable s_t drives conditional moments $\theta_m(s_t)$ and $\sigma_m(s_t)$
- s_t follows n-state continuous time Markov chain, $s_t \in \{1, \cdots, n\}$
- transition probabilities:

$$Pr(s_{t+\Delta} = j | s_t = i) \approx \lambda_{ij} \Delta$$

Chen (2008): Stochastic Discount Factor

- **Proposition**: The stochastic discount factor is:

$$\frac{dm_t}{m_t} = -r\left(s_t\right)dt - \eta\left(s_t\right)dW_t^m + \sum_{s_t \neq s_{t-}} \left(e^{\kappa\left(s_{t-},s_t\right)} - 1\right) dM_t^{\left(s_{t-},s_t\right)}$$

- $r =$ riskfree rate
- $\eta =$ risk price for systematic Brownian shock W_t^m

$$\eta(s) = \gamma\sigma_m(s)$$

- large shocks from $s_t \Rightarrow$ jumps in discount factor
- $\kappa\left(j,k\right) =$ relative jump size of discount factor when s switches from j to $k \Rightarrow$ risk prices for large shocks

Firms' Cash Flows

- A firm generates a perpetual stream of cash flows, X_t:

$$\frac{dX_t}{X_t} = \theta_X\left(s_t\right)dt + \sigma_{X,m}\left(s_t\right)dW_t^m + \sigma_f dW_t^f$$

with

$$\begin{aligned}\theta_X(s) &= a(\theta_m(s) - \overline{\theta}_m) + \overline{\theta}_X \\ \sigma_{X,m}(s) &= b(\sigma_m(s) - \overline{\sigma}_m) + \overline{\sigma}_{X,m}\end{aligned}$$

- $\overline{\theta}_m, \overline{\sigma}_m =$ average growth rate and volatility of consumption
- $\overline{\theta}_X, \overline{\sigma}_{X,m} =$ average growth rate and systematic volatility of the firm
- The rest is similar to Leland Leland (1994), Leland and Toft (1996), Duffien and Lando (2001) etc.
- That is, the firm issues debt, gets some tax shields from taxes and so on.
- The key is to determine now the optimal time of default, because in addition to cash flows X_t, now the pricing kernel is state dependent.

Chen (2008): Optimal Default

- Chen (2008) characterizes the value of equity and debt in each of the states.
- The key result is to show that there are n boundaries $X^1 < X^2 < .... < X^n$, each one corresponding to each state s^i, such that default occurs if $X_t \leq X_i$ in state i. The default boundaries are determined by a smooth pasting condition: The value of equity at the default boundary i satisfies and appropriate smooth pasting condition.
- Two types of defaults
  1. For given state, the firm loses cash and hits the boundary.
  2. There is a change in regime and the boundary suddenly increases, inducing default.

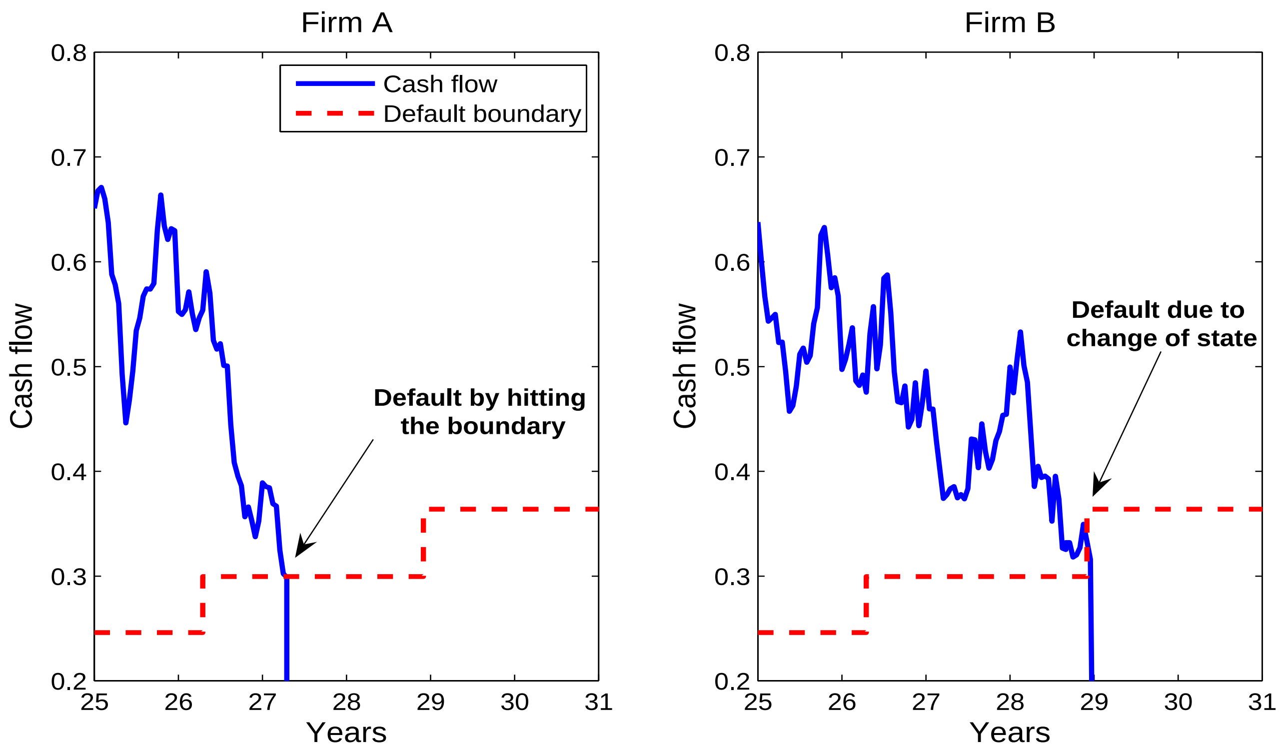


Figure 4: Illustration of Two Types of Defaults. In the left panel, default occurs when cash flow drops below a default boundary; in the right panel, default occurs when the default boundary jumps up, which is triggered by a change of the aggregate state.

What Determines the Default Boundaries?

- Optimal timing of default $\Leftarrow$ trade-off between the **value of liquidation** and the **value of continuation**
- Shareholders:
  - receive nothing back if liquidate
  - can keep firm alive if meet debt payments (may require additional capital)
  - in exchange, get claim on future dividends (cash flows net of debt payments), and retain the option to default
- Default Condition

$$value\, of\ future\, dividend\, payments + value\, of\ option\, to\, default\ < required\, contributions$$

Chen (2008): Results

- Chen (2008) calibrate model as follows
  - Pricing kernel is calibrated to mimic Bansal and Yaron (2004) parameters;
  - Taxes etc. to Grahm (2000).
  - Cash flow processes are calibrated to aggregate corporate profits. Idiosyncratic volatility is computed to match default frequencies.

Table V: Results For The Static Model

The table reports results of the static capital structure model, including the benchmark case (no business cycle variations), and the case with business cycle variations. It also reports 4 sets of comparative statics results: the first case lowers marginal tax rate when earnings are negative; the second adds equity issuance costs; the third combines the first two (the full model); the fourth case sets default loss α to a constant. Def10 - 10-year cumulative default probability; Rec - average recovery rate for firm's debt; VolRec - volatility of recovery rates; Spr10 - average credit spread for a 10-year coupon bond; Lev - market leverage; IntCov - Interest Coverage (Cash Flow/Coupon); TaxBen - Net tax benefits as measured by percentage increases in firm value; sprd - average credit spread of consol bond; ERx - exp. excess return on equity.

Panel A: Benchmark Case

	Def10	Rec	VolRec	Spr10	Lev	IntCov	TaxBen	Sprd	ERx
Baa	4.9%	48.0%	-	56.5	66.7%	0.7	10.8%	79.1	7.0%
Aaa	0.6%	48.0%	-	7.1	91.2%	0.1	20.0%	4.2	8.3%

Panel B: Model with Business-cycle Variation

	Def10	Rec	VolRec	Spr10	Lev	IntCov	TaxBen	Sprd	ERx
Baa	4.9%	45.3%	6.9%	141.3	50.4%	1.7	5.3%	262.8	9.3%
	(1.2%)	(1.8%)	(0.3%)	(9.7)	(3.5%)	(0.4)	(0.4%)	(50.5)	(1.6%)
Aaa	0.6%	46.5%	7.0%	43.4	52.2%	1.3	6.9%	81.4	6.6%
	(0.1%)	(1.6%)	(0.2%)	(1.1)	(1.8%)	(0.2)	(0.3%)	(17.1)	(1.2%)

Chen (2008): Default Clustering

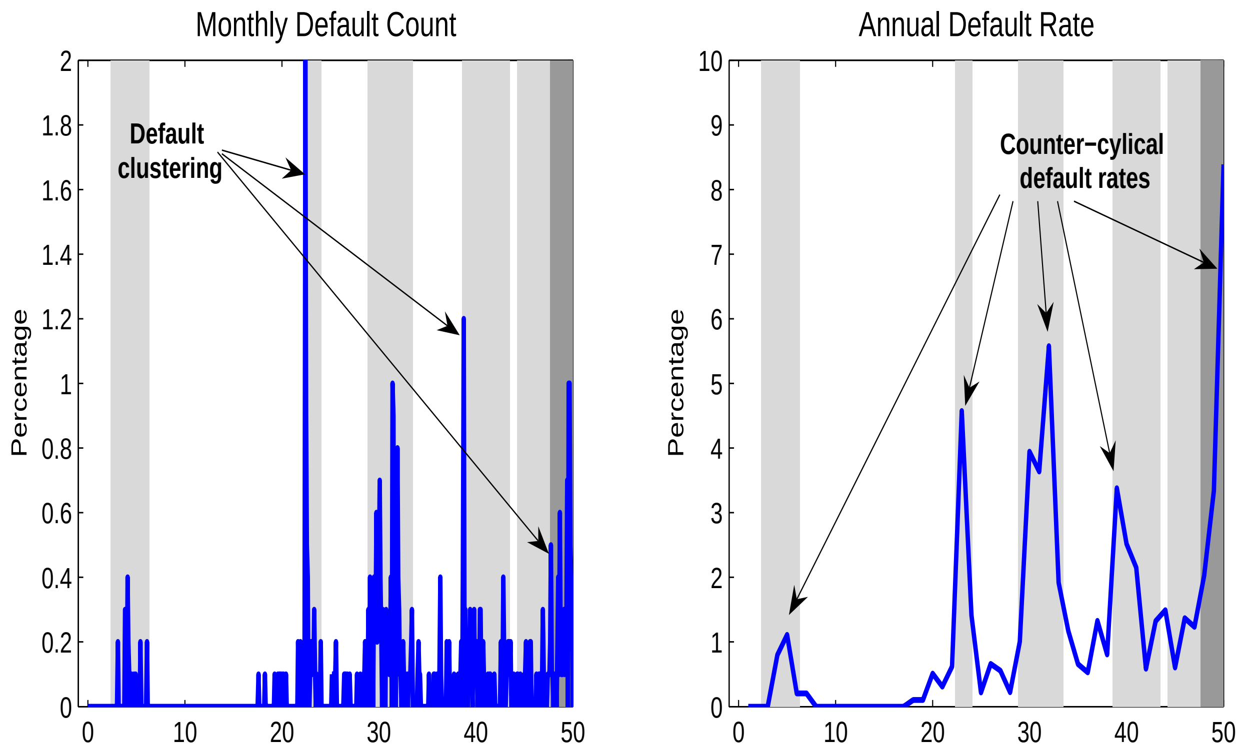


Figure 9: Simulated Default and the Annualized Default Rates. This figure plots the monthly default counts and annual default rates from a simulation of 1000 firms over 50 years. Areas with no shades are periods where the economy is in the state of high growth and median uncertainty. Light shades denote the state of low growth and median uncertainty. Dark shades denote the state of low growth and high uncertainty.

Conclusion

- Credit risk in dynamic macro-economic environment is interesting topic
- Structural models to default are kind of old, although still explored
- Seriously thinking about moral hazard and asymmetric information in these models seem important (e.g. extend the framework of Benmelech, Kandel and Veronesi (2008))
- The issue is always a quantitative one: how much can we explain of the facts?