Modern Dynamic Asset Pricing Models

Lecture Notes 4.

Term Structure Models

Pietro Veronesi

Graduate School of Business,
University of Chicago
CEPR, NBER

Outline

1. The building blocks of term structure models
   - No arbitrage
   - The fundamental pricing equation
   - The market price of risk and risk neutral pricing
2. Affine Models
   - Bond Pricing Formula
   - Estimation Issues
   - Predictability and Volatility
   - The failure of "completely affine models"
   - The rise of "effectively affine models"
3. Gaussian, Linear Quadratic Models
   - Empirical performance
4. Habit Formation and Time Varying Market Price of Risk

No Arbitrage and Term Structure Models

- Bond prices depend only on n factors $\mathbf{X}_t = (X_t^1, X_t^2, ..., X_t^n)'$
  - Example 1: Macro Factors
  - Example 2: Level, Slope and Curvature
- A zero coupon bond will have a price generically denoted by

$$Z\left(\mathbf{X}_t, t; T\right)$$

- Assume $\mathbf{X}_t$ follows the joint process

$$d\mathbf{X}_t = \mathbf{m}\left(\mathbf{X}_t, t\right) dt + \mathbf{s}\left(\mathbf{X}_t, t\right) d\mathbf{W}_t \tag{1}$$

  - where $\mathbf{W}_t$ is a n dimensional vector of Brownian motions.
- **Question:** What is the price of a zero coupon bond $Z\left(\mathbf{X}_t, t; T\right)$?
  - No arbitrage condition imposes a restriction on zero coupon bonds of various maturities.
  - These restrictions translate in particular bond pricing formulas, depending on the specification of the model (1)

The No Arbitrage Argument

- Suppose we are long $N_1 = 1$ bonds $Z(\mathbf{X}_t, t; T_1)$ and we would like to hedge the position. How can we do it?
- The long bond depends on n sources of risk (n Brownian Motions) $\Longrightarrow$ we need a set of n other securities (other bonds) to hedge all of the risk.
- Consider then the portfolio

$$\Pi_t = N_1 Z(\mathbf{X}_t, t; T_1) + \sum_{j=2}^{n+1} N_j Z(\mathbf{X}_t, t; T_j)$$

- What is the dynamics of the portfolio Π_t?
  - Ito's Lemma applied to each bond i, $Z_i = Z(\mathbf{X}_t, t; T_i)$, states

$$dZ_i = \mu_{Z,i} dt + \boldsymbol{\sigma}_{Z,i} d\mathbf{W}_t$$

  - where

$$\mu_{Z,i} = \frac{\partial Z_i}{\partial t} + \frac{\partial Z_i}{\partial \mathbf{X}} \mathbf{m}(\mathbf{X}_t, t) + \frac{1}{2} tr \left(\frac{\partial^2 Z_i}{\partial \mathbf{X} \partial \mathbf{X}'} \mathbf{s}(\mathbf{X}_t, t) \mathbf{s}(\mathbf{X}_t, t)' \right)$$

$$\boldsymbol{\sigma}_{Z,i} = \frac{\partial Z_i}{\partial \mathbf{X}} \mathbf{s}(\mathbf{X}_t, t)$$

A No Arbitrage Argument - 2

- We obtain

$$d\Pi_t = \sum_{j=1}^{n+1} N_j dZ_j = \mathbf{N}' \, \boldsymbol{\mu}_Z dt + \mathbf{N}' \, \boldsymbol{\sigma}_Z \, d\mathbf{W}_t$$

- Choose $\mathbf{N}$ so that last term drops out. I.e. impose

$$\mathbf{N}' \, \boldsymbol{\sigma}_Z = \mathbf{0} \tag{2}$$

  - There are n equations in n unknowns $N_2, ..., N_{n+1}$ (recall that $N_1 = 1$).

- Given (2), the last term is zero, and thus $d\Pi_t = \mathbf{N} \, \boldsymbol{\mu}_Z dt$ is risk free.

- $\Longrightarrow$ Impose no arbitrage $\Longrightarrow$ $\boxed{d\Pi_t = r_t \, \Pi_t dt}$

- Substitute on both sides the relevant expressions:

$$\mathbf{N}' \, (\boldsymbol{\mu}_Z - r\mathbf{Z}_t) = 0$$

- We do not have any degrees of freedom left. So, given (2), this condition is satisfied if and only if there is a n vector $\boldsymbol{\lambda}_t$ (possibly that also depends on $\mathbf{X}$ or t), such that

$$\underset{((n+1)\times 1)}{(\boldsymbol{\mu}_Z - r\mathbf{Z}_t)} = \underset{((n+1)\times n)}{\boldsymbol{\sigma}_Z} \underset{(n \times 1)}{\boldsymbol{\lambda}_t}$$

The Fundamental Pricing Equation

- That is, for each bond, we have

$$\mu_{Z,i} - r_t Z_i = \sum_{j=1}^{n} \lambda_t^j \sigma_{Z,i}^j$$

  - The bond return premium (=LHS) depends on loadings λ_t^j on the n sources of risk (BMs)

$$\lambda_t^j = \textbf{Market Price of Risk Factor} \quad j$$

  - Since this relation must hold for *any* bond (and in fact, any security that depends on $\mathbf{X}_t$), we can eliminate the subscript.

- The fundamental pricing equation is finally obtained by substituting back μ_Z and $\boldsymbol{\sigma}_Z$

$$r_t Z = \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial \mathbf{X}} \left[\mathbf{m} \left(\mathbf{X}_t, t \right) - \mathbf{s} \left(\mathbf{X}_t, t \right) \boldsymbol{\lambda}_t \right] + \frac{1}{2} tr \left(\frac{\partial^2 Z}{\partial \mathbf{X} \partial \mathbf{X}'} \mathbf{s} \left(\mathbf{X}_t, t \right) \mathbf{s} \left(\mathbf{X}_t, t \right)' \right) \tag{3}$$

- To summarize, the price of the zero coupon bond $Z(\mathbf{X}_t, t; T)$ is the $\textbf{solution}$ to the Partial Differential Equation (3), subject to the final condition

$$Z \left(\mathbf{X}_T, T; T \right) = 1$$

Risk Neutral Pricing

- Finally, how can we "solve" the PDE?

1. Analytically, if possible: Some of these PDEs have closed form solutions (e.g. Affine Models)
2. Numerically: Buy a PDE solver for the computer, and let the computer crunch out the solution.
3. Apply Feynman Kac Theorem to the PDE (3)
   - Define
   $$\boldsymbol{\mu}\left(\mathbf{X}_t, t\right) = \mathbf{m}\left(\mathbf{X}_t, t\right) - \mathbf{s}\left(\mathbf{X}_t, t\right)\boldsymbol{\lambda}_t \tag{4}$$
   - The solution to the PDE is given by the Feynman Kac Formula:
   $$Z\left(\mathbf{X}_t, t; T\right) = E^Q\left[e^{-\int_t^T r_s ds} \times 1 | \mathbf{X}_t\right]$$
   - where $E^Q[\cdot]$ denotes expectation with respect to the modified factor process
   $$d\mathbf{X}_t = \boldsymbol{\mu}\left(\mathbf{X}_t, t\right) dt + \mathbf{s}\left(\mathbf{X}_t, t\right) d\mathbf{W}_t^Q \tag{5}$$
   - The process (5) is called **risk neutral** process.
     - The drift rate of the original process $\mathbf{m}$ has been adjusted for the market price of risk $\boldsymbol{\lambda}_t$, which, recall, determines the risk-return characteristics of risky bonds.

Risk Neutral Process and Risk Natural Process

- The distinction between the Risk Neutral and the Risk Natural process is only in the drift.

$$\begin{aligned} \text{Risk Natural} \Longrightarrow \quad d\mathbf{X}_t &= \mathbf{m}(\mathbf{X}_t, t)\, dt + \mathbf{s}(\mathbf{X}_t, t)\, d\mathbf{W}_t \\ \text{Risk Neutral} \Longrightarrow \quad d\mathbf{X}_t &= (\mathbf{m}(\mathbf{X}_t, t) - \mathbf{s}(\mathbf{X}_t, t)\,\boldsymbol{\lambda}_t)\, dt + \mathbf{s}(\mathbf{X}_t, t)\, d\mathbf{W}_t^Q \end{aligned}$$

- We can pass from risk natural to risk neutral by defining the new Brownian Motion

$$d\mathbf{W}_t^Q = d\mathbf{W}_t + \boldsymbol{\lambda}_t$$

- The market price of risk defines a new probability measure over which we take expectations.
  - Example: We saw that a simple model with power utility and expected inflation implies

$$\begin{aligned} \text{nominal rate} \quad dr &= (\widetilde{\alpha} - \beta r_t)\, dt + \sigma_i dW_t \\ \text{market price of risk} \quad \lambda &= \gamma \sigma_y \rho_{iy} + \sigma_q \rho_{iq} \end{aligned}$$

  - Given this information, the price of nominal bonds is determined as follows:
    1. Define the risk neutral process: $dr = (\widetilde{\alpha} - \beta r_t - \sigma_i \lambda)\, dt + \sigma_i dW_t^Q$
    2. Compute the price as $Z(r, t; T) = E_t^Q \left[e^{-\int_t^T r_u du} \right]$
    3. This can be accomplished, *equivalently*, by solving the PDE

$$rZ = \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r} (\widetilde{\alpha} - \beta r_t - \sigma_i \lambda) + \frac{1}{2} \frac{\partial Z^2}{\partial r^2} \sigma_i^2$$

Affine Models

- Affine term structure models are a good example of the pricing methodology.
- The term "Affine" stems from the fact that everything we discussed is affine (= linear + constant) in the risk neutral dynamics:

$$\begin{array}{rrcl} \text{risk free rate} & r_t & = & \delta_0 + \boldsymbol{\delta}_1' \mathbf{X}_t \\ \text{risk neutral drift} & \boldsymbol{\mu}(\mathbf{X}_t, t) & = & \widetilde{\mathcal{K}}\left(\widetilde{\boldsymbol{\theta}} - \mathbf{X}_t\right) \\ \text{diffusion term} & \mathbf{s}(\mathbf{X}_t, t) & = & \boldsymbol{\Sigma}\sqrt{\mathbf{S}_t} \end{array}$$

- where $\mathbf{S}_t$ is a diagonal matrix, with diagonal element

$$[\mathbf{S}_t]_{ii} = \alpha_i + \boldsymbol{\beta}_{\mathbf{i}}{}' \mathbf{X}_t$$

- The price of the bond is then given by the usual formula $Z(\mathbf{X}_t, t; T) = E^Q\left[e^{-\int_t^T r_s ds}\right]$
- Or, equivalently, it satisfies the PDE

$$rZ = \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial \mathbf{X}} \widetilde{\mathcal{K}}\left(\widetilde{\boldsymbol{\theta}} - \mathbf{X}_t\right) + \frac{1}{2} tr\left(\frac{\partial^2 Z}{\partial \mathbf{X} \partial \mathbf{X}'} \boldsymbol{\Sigma} \mathbf{S}_t \boldsymbol{\Sigma}'\right)$$

Solving for Bond Prices

- How do we solve for bond prices?
  - Either solve the expectation or the PDE. For Affine Models, the PDE is "simpler".
- **Method of Undetermined Coefficients** (sketch)
  1. Conjecture: $Z(\mathbf{X}, t; T) = e^{A(t;T) - \mathbf{B}(t;T)'\mathbf{X}_t}$
  2. Compute derivatives:

$$\frac{\partial Z}{\partial t} = \left(\frac{\partial A}{\partial t} - \frac{\partial \mathbf{B}'}{\partial t}\mathbf{X}_t\right) Z; \qquad \frac{\partial Z}{\partial \mathbf{X}} = -\mathbf{B}(t;T)Z; \qquad \frac{\partial^2 Z}{\partial \mathbf{X}\partial \mathbf{X}'} = \mathbf{B}(t;T)\mathbf{B}(t;T)'Z$$

  3. Substitute r and partial derivatives in PDE, and divide by Z

$$\underbrace{\delta_0 + \boldsymbol{\delta}_1'\mathbf{X}_t}_{r_t} = \underbrace{\left(\frac{\partial A}{\partial t} - \frac{\partial \mathbf{B}'}{\partial t}\mathbf{X}_t\right)}_{\frac{\partial Z}{\partial t}} \underbrace{-\mathbf{B}(t;T)'\widetilde{\mathcal{K}}\left(\widetilde{\boldsymbol{\theta}} - \mathbf{X}_t\right)}_{\frac{\partial Z}{\partial \mathbf{X}}\boldsymbol{\mu}(\mathbf{X}_t)} + \frac{1}{2}\underbrace{\sum_{i=1}^{n}[\boldsymbol{\Sigma}'\mathbf{B}(t;T)]_{ii}\left(\alpha_i + \boldsymbol{\beta}_i'\mathbf{X}_\mathbf{t}\right)}_{tr\left(\frac{\partial^2 Z}{\partial \mathbf{X}\partial \mathbf{X}'}\boldsymbol{\Sigma}\mathbf{S}_t\boldsymbol{\Sigma}'\right)}$$

Solving for Bond Prices

4. Collect terms and obtain two Ordinary Differential Equations:

$$\underbrace{\left(\frac{\partial \mathbf{B}'}{\partial t} + \boldsymbol{\delta}_1' - \mathbf{B}(t;T)'\widetilde{\mathcal{K}} - \frac{1}{2}\sum_{i=1}^{n}[\boldsymbol{\Sigma}'\mathbf{B}(t;T)]_{ii}\boldsymbol{\beta}_i'\right)}_{=0}\mathbf{X}_t = \underbrace{\frac{\partial A}{\partial t} - \delta_0 - \mathbf{B}(t;T)'\widetilde{\mathcal{K}}\widetilde{\boldsymbol{\theta}} + \frac{1}{2}\sum_{i=1}^{n}[\boldsymbol{\Sigma}'\mathbf{B}(t;T)]_{ii}\alpha_i}_{=0}$$

– with final condition $A(T;T) = 0$ and $\mathbf{B}(T;T) = \mathbf{0}$

5. Solve the ODEs (much easier than solving the PDE)
   – From $t = T$ we have the conditions $\mathbf{B}(T;T) = 0$ and $A(T;T) = 0$
   – Then work backwards, by discretizing time: Since

$$\frac{\partial \mathbf{B}(t;T)}{\partial t} \approx \frac{\mathbf{B}(t;T) - \mathbf{B}(t-dt;T)}{dt}; \qquad \frac{\partial A(t;T)}{\partial t} \approx \frac{A(t;T) - A(t-dt;T)}{dt}$$

   – we can write

$$\begin{aligned}
\mathbf{B}'(t-dt;T) &= \mathbf{B}'(t;T) + \left(\boldsymbol{\delta}_1' - \mathbf{B}(t;T)'\widetilde{\mathcal{K}} - \frac{1}{2}\sum_{i=1}^{n}[\boldsymbol{\Sigma}'\mathbf{B}(t;T)]_{ii}\boldsymbol{\beta}_i'\right)dt \\
A(t-dt;T) &= A(t;T) + \left(-\delta_0 - \mathbf{B}(t;T)'\widetilde{\mathcal{K}}\widetilde{\boldsymbol{\theta}} + \frac{1}{2}\sum_{i=1}^{n}[\boldsymbol{\Sigma}'\mathbf{B}(t;T)]_{ii}\alpha_i\right)dt
\end{aligned}$$

Affine Term Structure Models

- Model encompasses many different models. For instance:
  1. Vasicek: $dr_t = \tilde{k}(\tilde{\theta} - r_t)dt + sdW_t$
  2. Cox Ingersoll and Ross: $dr_t = \tilde{k}(\tilde{\theta} - r_t)dt + \sqrt{\alpha r_t}dW_t$
  3. Fong and Vasicek (Stochastic Volatility):

$$\begin{aligned} dr_t &= \tilde{k}_r\left(\tilde{\theta}_r - r_t\right)dt + \sqrt{v_t}dW_{1,t} \\ dv_t &= \tilde{k}_v\left(\tilde{\theta}_v - v_t\right)dt + \sigma\sqrt{v_t}dW_{2,t} \end{aligned}$$

  4. Canonical $A_2(3)$ model:
     - Largely used model with 3 factors, with 2 affecting volatility.

$$\begin{aligned} dr_t &= \tilde{k}_r\left(\theta_t - r_t\right)dt + \sqrt{v_t}dW_{1,t} \\ dv_t &= \tilde{k}_v\left(\tilde{\theta}_v - v_t\right)dt + \sigma_v\sqrt{v_t}dW_{2,t} \\ d\theta_t &= \tilde{k}_\theta\left(\tilde{\theta}_\theta - \theta_t\right)dt + \sigma_\theta\sqrt{\theta_t}dW_{3,t} \end{aligned}$$

- All of these models have the same form of the price of bonds: $Z(\mathbf{X}_t, \tau) = e^{A(\tau) - \mathbf{B}(\tau)'\mathbf{X}_t}$
  - where $\tau = T - t =$ maturity, and $A(\tau) = A(0;\tau)$, $\mathbf{B}(\tau) = \mathbf{B}(0;\tau)$.
- There are technical issues: Not all of the parametrizations are feasible.
  - We must guarantee volatility is always positive, for instance. Dai and Singleton (2000) provide canonical representation with parameter restrictions.

Estimation and the Market Price of Risk

- To estimate the model we have to use
  1. The cross-sectional information of bond prices (yields) $\Longrightarrow$ Need of Risk Neutral Model for Pricing
  2. The time-series information of bond prices (yields) $\Longrightarrow$ Need the Risk *Natural* Model to describe the true time series dynamics of factors
- To use both types of information, we need a specification of the market price of risk $\boldsymbol{\lambda}_t$
  - The specification of the market price of risk has been a hot topic of research in recent years.
- The most classic specification of the market price of risk (e.g. Dai and Singleton (2000)) has

$$\boldsymbol{\lambda}_t = \sqrt{\mathbf{S}_t}\boldsymbol{\lambda}_1 \tag{6}$$

  - This is convenient: The risk *natural* drift is also affine:

$$\mathbf{m}(\mathbf{X}_t) \ = \ \widetilde{\mathcal{K}}\left(\widetilde{\boldsymbol{\theta}} - \mathbf{X}_t\right) + \left(\boldsymbol{\Sigma}\sqrt{\mathbf{S}_t}\right)\left(\sqrt{\mathbf{S}_t}\boldsymbol{\lambda}_1\right) = \mathcal{K}\left(\boldsymbol{\theta} - \mathbf{X}_t\right)$$

- We will see that specification (6) however yields counterfactual implications.

Estimation Issues

- Hard problem, as the conditional distribution of $\mathbf{X}_{t+1}$ is hard to compute for general specifications.
- Many methodologies overcome the problem
  - Simulation-based Efficient Method of Moments (e.g. Dai and Singleton (2000))
  - Characteristic function computation (Singleton (2001))
  - Quasi-Maximum Likelihood Estimation (very popular, as it is the simplest)
- Methodology: First, back out the factors $\mathbf{X}_t$
  - The yields from affine model is *affine* in factors (yet again)

$$Y_t(\tau) = -\frac{\log\left(Z(\mathbf{X}_t, \tau)\right)}{\tau} = -\frac{A(\tau)}{\tau} + \frac{\mathbf{B}(\boldsymbol{\tau})'}{\tau}\mathbf{X}_t$$

  - Assume that exactly n yields are observed perfectly: τ_1, ...τ_n. Let $\mathbf{Y}_t = [Y_t(\tau_1), ..., Y_t(\tau_n)]'$
  - Defining $\mathbf{H}_0$ and $\mathbf{H}_1$ in the obvious way from the above relation, we have

$$\mathbf{Y}_t = \mathbf{H}_0 + \mathbf{H}_1\mathbf{X}_t$$

  - Given the parameters of the model Θ, invert the relation to obtain

$$\widehat{\mathbf{X}}_t = \mathbf{H}_1^{-}1\left(\mathbf{Y}_t - \mathbf{H}_0\right)$$

Estimation Issues - 2

- Second, assume all of the other yields are estimated with error.
  - Let $\widehat{\mathbf{Y}}_t$ be vectors of yields that are imperfectly measured. Given $\widehat{\mathbf{X}}_t$, the errors are

$$\boldsymbol{\varepsilon}_t = \mathbf{Y}_t^{data} - \widehat{\mathbf{Y}}_t$$

  - Assume errors are i.i.d. normal, cross-sectionally uncorrelated.
- Third, compute the likelihood function or the moments of interest.
  - In QML, assume $\mathbf{X}_{t+1}$ conditional on $\mathbf{X}_t$ is normally distributed.
    - The mean and variance are known (see appendix Duffee (2002))

$$\begin{aligned} E[\mathbf{X}_T|\mathbf{X}_t] &= \left(\mathbf{I} - e^{-\mathcal{K}(T-t)}\right)\boldsymbol{\theta} + e^{-\mathcal{K}(T-t)}\mathbf{X}_t \\ V[\mathbf{X}_T|\mathbf{X}_t] &= \mathbf{N}\mathbf{b_0}\mathbf{N}' + \sum_{i=1}^{n}\left(\mathbf{N}\mathbf{b_j}\mathbf{N}'\mathbf{N}_{j,i}^{-1}\right) \end{aligned}$$

    - where all of the quantities are matrices known in closed form.
  - From here, the conditional distribution of $\mathbf{Y_{t+1}}$ is obtained from the change in variable

$$f_Y(\mathbf{Y}_{t+1}|\mathbf{Y}_t) = \frac{1}{|det(\mathbf{H}_1|} f_X(\mathbf{X}_{t+1}|\mathbf{X}_t)$$

  - Add the likelihood for the observation errors, and we are done.

Predictability of Bond Returns and Affine Models

- We saw that bond returns are strongly predictable.
- Moreover, the predictability seems to stem mainly from the slope of the term structure.
  - In particular, the strong volatility of yields does not seem to predict return at all.
- Duffee (2002) reports the following regression results, confirming the empirical findings in the earlier lecture.

Table I

Regressions of Excess Returns to Treasury Bonds, July 1961 through December 1998

Monthly excess returns to portfolios of Treasury bonds are regressed on the previous month's term-structure slope and an estimate of the interest rate volatility during the previous month. The slope of the term structure is measured by the difference between five-year and three-month zero-coupon yields (interpolated from coupon bonds). Monthly volatility is measured by the square root of the sum of squared daily changes in the five-year zero-coupon bond yield. Asymptotic t-statistics, adjusted for generalized heteroskedasticity, are in parentheses. There are 449 monthly observations.

Maturity (years)	Mean Excess Return (%)	Coefficient on Slope	Coefficient on Volatility	Std. Dev. of Fitted Excess Rets
$0 < m \leq 1$	0.011	0.027	0.116	0.036
		(1.76)	(0.96)	
$1 < m \leq 2$	0.045	0.085	0.413	0.119
		(1.85)	(1.27)	
$2 < m \leq 3$	0.064	0.132	0.582	0.179
		(1.88)	(1.20)	
$3 < m \leq 4$	0.074	0.187	0.706	0.241
		(2.38)	(1.35)	
$4 < m \leq 5$	0.063	0.214	0.692	0.265
		(2.37)	(1.16)	
$5 < m \leq 10$	0.094	0.296	0.804	0.354
		(2.69)	(1.08)	

Source: Duffee (2002)

Predictability of Bond Returns and Affine Models

- What is the expected return of bond returns?
  - As discussed earlier, the premium is given by
  $$E[dZ - rZ] = E\left[\mu_{Z,i} - r_t Z_i\right] = \boldsymbol{\sigma}_{Z,i}\boldsymbol{\lambda}_t$$
  - From the affine structure, the diffusion term is
  $$\boldsymbol{\sigma}_{Z,i} = \frac{\partial Z_i}{\partial \mathbf{X}_t}\,\mathbf{s}(\mathbf{X}_t) = -Z_i \mathbf{B}(\tau)'\boldsymbol{\Sigma}\sqrt{\mathbf{S}_t}$$
  - The percentage expected return is then given by
  $$\text{Expected Excess Return} = E_t\left[\frac{dZ}{Z} - r\right] = -\mathbf{B}(\tau)'\boldsymbol{\Sigma}\sqrt{\mathbf{S}_t}\boldsymbol{\lambda}_t$$
  - Using the traditional market price of risk, we obtain
  $$E_t\left[\frac{dZ}{Z} - rdt\right] = -\mathbf{B}(\tau)'\boldsymbol{\Sigma}\mathbf{S}_t\boldsymbol{\lambda}_1 = \sum_{i=1}^{n} a_i(\tau) + \mathbf{b}_i(\tau)\mathbf{X}_t$$
- The key insight is that $b_{i,j}(\tau)$ are different from zero (and thus time varying) *only* for those factors $X_{j,t}$ that affect the volatility of $\mathbf{X}_t$, and thus the volatility of bond prices
- $\Longrightarrow$ Completely affine model imply a tight relation between expected returns and bond volatility.

Duffee (2002) Evidence on Affine Models

- Duffee (2002) estimates a standard affine model ($A_2(3)$) and shows that it does not forecast future yields.

Comparison of In-sample Forecasting Performance (RMSE)

				Unrestricted			Preferred		
				C. A.	E. A.		C. A.	E. A.	
Bond Maturity	Forecast Horizon	RW	OLS	$A_2(3)$	$A_0(3)$	$A_1(3)$	$A_2(3)$	$A_0(3)$	$A_1(3)$
6 months	3	1.023	1.020	1.045	1.009	1.019	1.048	1.009	1.019
2 years	3	0.871	0.869	0.880	0.837	0.847	0.883	0.837	0.853
10 years	3	0.549	0.532	0.554	0.526	0.543	0.554	0.528	0.547
6 months	6	1.376	1.370	1.418	1.342	1.367	1.427	1.345	1.368
2 years	6	1.154	1.149	1.173	1.091	1.121	1.181	1.089	1.133
10 years	6	0.760	0.722	0.774	0.711	0.756	0.772	0.713	0.764
6 months	12	1.803	1.797	1.843	1.731	1.798	1.868	1.742	1.798
2 years	12	1.541	1.529	1.566	1.450	1.527	1.583	1.445	1.544
10 years	12	1.109	1.018	1.137	1.011	1.121	1.131	1.009	1.133

Source: Duffee (2002)

Duffee (2002) Evidence on Affine Models

- Pricing errors in affine models are strongly related to slope, showing that even a 3 factor affine model with time varying variance fails to capture independent variation in slope.

The Relation Between In-sample Forecast Errors and the Yield-curve Slope

			Unrestricted			Preferred		
			C. A.	E. A.		C. A.	E. A.	
Bond Maturity	Forecast Horizon	RW	$A_2(3)$	$A_0(3)$	$A_1(3)$	$A_2(3)$	$A_0(3)$	$A_1(3)$
6 months	3	0.072	−0.182	−0.041	−0.135	−0.182	0.019	−0.124
		(0.73)	(−1.84)	(−0.42)	(−1.39)	(−1.83)	(0.19)	(−1.27)
2 years	3	−0.043	−0.182	−0.043	−0.134	−0.183	0.013	−0.129
		(−0.52)	(−2.22)	(−0.54)	(−1.69)	(−2.22)	(0.16)	(−1.63)
10 years	3	−0.125	−0.159	−0.027	−0.141	−0.158	−0.018	−0.140
		(−2.72)	(−3.50)	(−0.61)	(−3.19)	(−3.49)	(−0.39)	(−3.15)
6 months	6	0.118	−0.324	−0.085	−0.252	−0.326	0.015	−0.233
		(0.91)	(−2.55)	(−0.69)	(−2.03)	(−2.53)	(0.12)	(−1.88)
2 years	6	−0.082	−0.326	−0.091	−0.261	−0.330	−0.003	−0.249
		(−0.76)	(−3.17)	(−0.91)	(−2.60)	(−3.17)	(−0.03)	(−2.51)
10 years	6	−0.220	−0.280	−0.049	−0.255	−0.280	−0.031	−0.252
		(−3.45)	(−4.48)	(−0.78)	(−4.14)	(−4.46)	(−0.50)	(−4.09)
6 months	12	0.129	−0.567	−0.208	−0.484	−0.575	−0.058	−0.453
		(0.70)	(−3.30)	(−1.21)	(−2.74)	(−3.30)	(−0.33)	(−2.60)
2 years	12	−0.158	−0.551	−0.191	−0.486	−0.560	−0.069	−0.462
		(−1.06)	(−3.86)	(−1.32)	(−3.26)	(−3.86)	(−0.48)	(−3.15)
10 years	12	−0.410	−0.506	−0.135	−0.480	−0.507	−0.101	−0.472
		(−3.62)	(−4.53)	(−1.22)	(−4.24)	(−4.52)	(−0.92)	(−4.19)

The Market Price of Risk

- Why the traditional specification of the market price of risk has the form $\boldsymbol{\lambda}_t = \sqrt{\mathbf{S}_t}\boldsymbol{\lambda}_1$?
  - Historical accident? Symmetry? (With this assumption, risk neutral and risk natural model are both affine)
- Other specifications of the market price of risk may lead to better behavior of returns in the time series
  - Recall that this specification does not affect the risk neutral dynamics, and thus the pricing formula remains identical.
- Duffee (2002) and Duarte (2003) propose generalizations to the market price of risk.
  - The key is to delink the volatility of yields from expected returns.

Essentially Affine Models

- Duffee (2002), in particular, proposes the "essentially affine model"

$$\boldsymbol{\lambda}_t = \mathbf{S}_t^{1/2}\boldsymbol{\lambda}_1 + \widehat{\mathbf{S}}_t^{-1/2}\boldsymbol{\lambda}_2\mathbf{X}_t$$

  - where $\boldsymbol{\lambda}_2$ is a$n \times n$ matrix, and $\widehat{\mathbf{S}}_t$ is a diagonal matrix such that

$$[\widehat{\mathbf{S}}_t]_{ii} = \begin{cases} \alpha_i + \boldsymbol{\beta}_{\mathbf{i}}\mathbf{X}_t & \text{if} \quad \min(\alpha_i + \beta_i\mathbf{X}_t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Under the physical measure, the model is still affine. In fact the *risk natural drift* is now given by

$$\begin{aligned} \mathbf{m}(\mathbf{X}_t) &= \widetilde{\mathcal{K}}\left(\widetilde{\boldsymbol{\theta}} - \mathbf{X}_t\right) + \boldsymbol{\Sigma}\mathbf{S}_t^{\frac{1}{2}}\left(\mathbf{S}_t^{\frac{1}{2}}\boldsymbol{\lambda}_1 + \widehat{\mathbf{S}}_t^{-\frac{1}{2}}\boldsymbol{\lambda}_2\mathbf{X}_t\right) \\ &= \mathcal{K}\boldsymbol{\theta} - \mathcal{K}\mathbf{X}_t \end{aligned}$$

- where now

$$\begin{aligned} \mathcal{K} &= \mathcal{K}^Q - \boldsymbol{\Sigma} diag(\boldsymbol{\lambda}_1)\boldsymbol{\beta} + \boldsymbol{\Sigma}\hat{\mathbf{I}}\boldsymbol{\lambda}_2 \\ \mathcal{K}\boldsymbol{\theta} &= \widetilde{\mathcal{K}}\widetilde{\boldsymbol{\theta}} + \boldsymbol{\Sigma} diag(\boldsymbol{\lambda}_1)\boldsymbol{\alpha} \end{aligned}$$

- and $\hat{\mathbf{I}}$ is the identity matrix, but with zeros wherever $[\widehat{\mathbf{S}}_t]_{[ii]} = 0$

What is the Gist of Essentially Affine Models?

- It is useful to consider an example. Consider the Vasicek model

$$dr = k_r \left(\overline{r} - r_t\right) dt + \sigma_r dW_{r,t}$$

  – The market price of risk under the standard setting is constant λ, thus RN process is simply

$$dr = k_r \left(\overline{r}^Q - r_t\right) dt + \sigma_r dW_{r,t}^Q$$

  – The model is the same as in previous Lecture. In particular, recall, it implies constant volatility of bond returns and no predictability.

- Consider a new factor f_t, following the square root process:

$$df = k_f \left(\overline{f} - f_t\right) dt + \sigma_f \sqrt{f} dW_{f,t}$$

  – In the Vasicek model with the traditional market price of risk, this factor would have no impact on bond prices.
  – However, within the more general framework of "essentially affine models", we can specify a market price of risk so that bond prices will depend on f_t as well.

A Risk Factor in Vasicek Model

- Given $\mathbf{X}_t = (r_t, f_t)'$, the joint process has diffusion

$$\mathbf{s}(X) = \mathbf{\Sigma}\mathbf{S}_t^{\frac{1}{2}} = \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & f_t \end{pmatrix}^{\frac{1}{2}}$$

- Assume now the essentially affine market price of risk

$$\lambda_t = S_t^{\frac{1}{2}}\lambda_1 + \widehat{S}_t^{-\frac{1}{2}}\lambda_2\mathbf{X}_t$$

- Since f_t can reach zero, we have

$$\mathbf{S}_t = \begin{pmatrix} 1 & 0 \\ 0 & f_t \end{pmatrix} \Longrightarrow \widehat{\mathbf{S}}_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- The risk neutral drift then is given by

$$\begin{aligned}
\boldsymbol{\mu}(r_t, f_t) &= \mathbf{m}(r_t, f_t) - \mathbf{s}(\mathbf{X})\boldsymbol{\lambda}_t \\
&= \mathbf{m} - \mathbf{\Sigma}\mathbf{S}_t^{\frac{1}{2}}\left(\mathbf{S}_t^{\frac{1}{2}}\boldsymbol{\lambda}_1 + \widehat{\mathbf{S}}_t^{-\frac{1}{2}}\boldsymbol{\lambda}_2\mathbf{X}_t\right) \\
&= \mathbf{m} - \begin{pmatrix} \sigma_r\lambda_{1,1} \\ \sigma_f\lambda_{1,2}f_t \end{pmatrix} - \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda_{2,11} & \lambda_{2,12} \\ \lambda_{2,21} & \lambda_{2,22} \end{pmatrix}\mathbf{X}_t\right) \\
&= \mathbf{m} - \begin{pmatrix} \sigma_r\lambda_{1,1} \\ \sigma_f\lambda_{1,2}f_t \end{pmatrix} - \begin{pmatrix} \lambda_{2,11} & \lambda_{2,12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r_t \\ f_t \end{pmatrix}
\end{aligned}$$

The Factor Processes and Bond Price

- Some algebra shows

$$\begin{aligned} dr_t &= \left(\tilde{k}_r - \tilde{k}_{rr} r_t - \tilde{k}_{rf} f_t\right) dt + \sigma_r dW_{r,t} \\ df_t &= \left(\tilde{k}_f - \tilde{k}_{ff} f_t\right) dt + \sigma_f \sqrt{f_t} dW_{f,t} \end{aligned}$$

- where

$$\begin{aligned} \tilde{k}_r &= k_r \overline{r} - \sigma_r \lambda_{1,1}; \quad \tilde{k}_{rr} = \left(k_r + \lambda_{2,11}\right); \quad \tilde{k}_{rf} = \lambda_{2,12} \\ \tilde{k}_f &= k_f \overline{f}; \quad k_{ff} = \left(k_f + \sigma_f \lambda_{1,2}\right) \end{aligned}$$

- This is an affine model of the term structure, so bond prices will be

$$Z\left(r_t, f_t, \tau\right) = e^{A(\tau) - B_r(\tau) r_t - B_f(\tau) f_t}$$

- where $A\left(\tau\right), B_r\left(\tau\right), B_f\left(\tau\right)$ satisfy some ODEs.
- Note, then, that the long term bond yield depends on f while, under the physical measure, the short term rate r does not (as it follows a Vasicek Model).

$$y_t\left(\tau\right) = -\frac{A\left(\tau\right)}{\tau} + \frac{B_r\left(\tau\right)}{\tau} r_t + \frac{B_f\left(\tau\right)}{\tau} f_t$$

Expected Return

- The process for bonds (under the physical measure) is then given by

$$\frac{dZ}{Z} = (r_t + \mu_Z)dt + \boldsymbol{\sigma}_Z d\mathbf{W}_t$$

- The diffusion term depends on the factor f_t

$$\boldsymbol{\sigma}_Z = \left[\frac{1}{Z}\frac{\partial Z}{\partial r}, \frac{1}{Z}\frac{\partial Z}{\partial f}\right] \boldsymbol{\Sigma}\mathbf{S}_t^{\frac{1}{2}} = -\left[B_r(\tau)\,\sigma_r, B_f(\tau)\,\sigma_f\sqrt{f_t}\right]$$

- Expected Return is given by

$$\begin{aligned} E\left[\frac{dZ}{Z}\right] - r_t = \mu_Z &= \boldsymbol{\sigma}_Z \boldsymbol{\lambda}_t = \left(\frac{1}{Z}\frac{\partial Z}{\partial r}, \frac{1}{Z}\frac{\partial Z}{\partial f}\right) \Sigma\mathbf{S}_t^{\frac{1}{2}} \left(\mathbf{S}_t^{\frac{1}{2}}\boldsymbol{\lambda}_1 + \widehat{\mathbf{S}}_t^{-\frac{1}{2}}\boldsymbol{\lambda}_2\mathbf{X}_t\right) \\ &= -\left\{B_r(\tau)\,\sigma_r\lambda_{1,1}\right\} - \left\{B_r(\tau)\,\lambda_{2,11}\right\} r_t - \left\{B_r(\tau)\,\lambda_{2,12} + B_f(\tau)\,\sigma_f\lambda_{1,2}\right\} f_t \end{aligned}$$

- Depending on market prices $\lambda_1' s$ and $\lambda_2' s$, the expected return not only is time varying, but it can even change sign.

- In particular, it is no longer the case that volatility of bond returns is so tightly linked to expected return

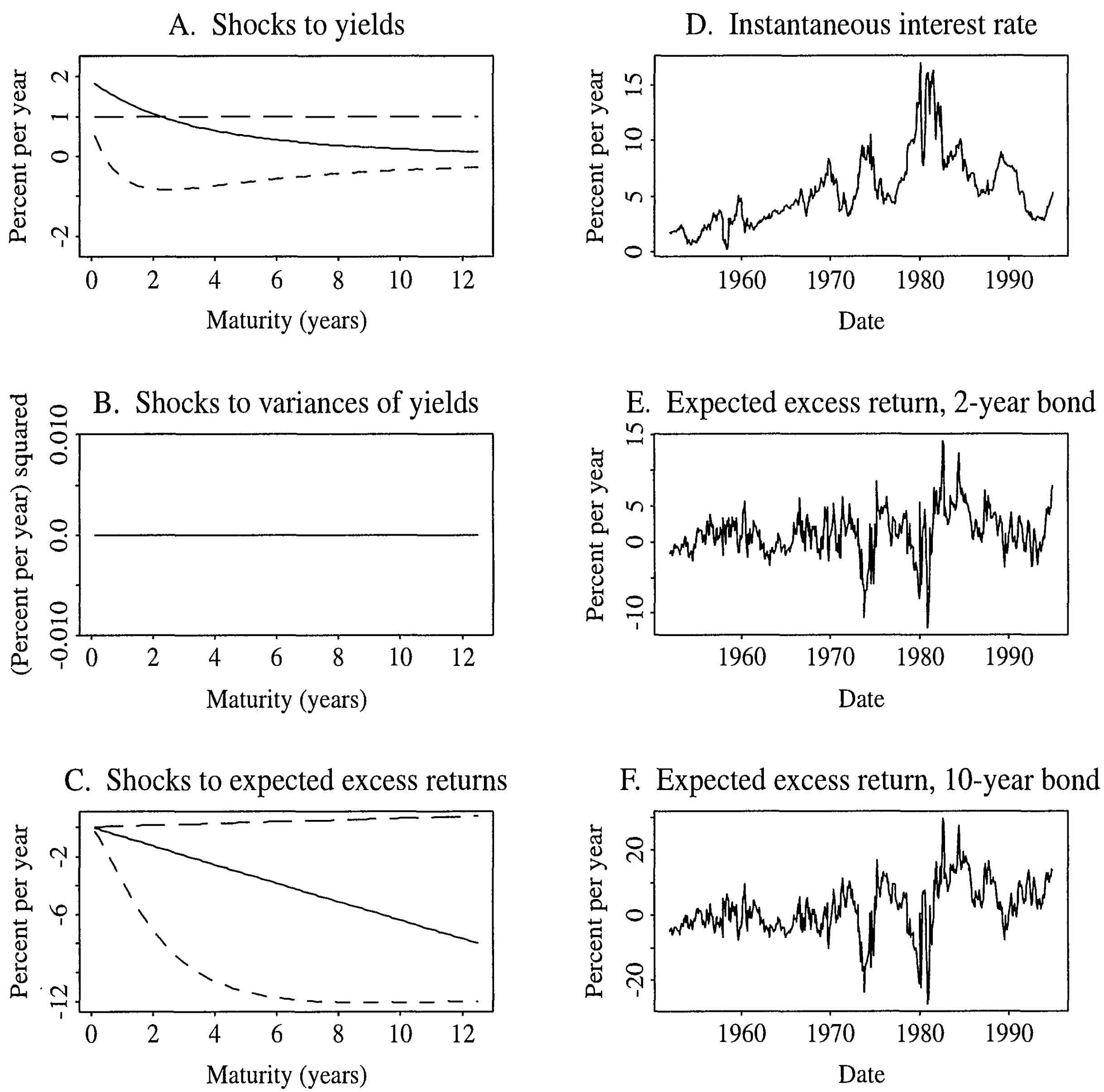
The Performance of Essentially Affine Models

Comparison of Out-of-Sample Forecasting Performance (RMSE)

				Unrestricted			Preferred		
				C. A.	E. A.		C. A.	E. A.	
Bond Maturity	Forecast Horizon	RW	OLS	$A_2(3)$	$A_0(3)$	$A_1(3)$	$A_2(3)$	$A_0(3)$	$A_1(3)$
6 months	3	0.298	0.298	0.325	0.281	0.288	0.350	0.281	0.284
2 years	3	0.499	0.511	0.501	0.454	0.458	0.523	0.457	0.450
10 years	3	0.484	0.498	0.476	0.460	0.457	0.485	0.469	0.453
6 months	6	0.400	0.413	0.483	0.373	0.399	0.548	0.365	0.385
2 years	6	0.652	0.675	0.656	0.565	0.576	0.711	0.566	0.560
10 years	6	0.669	0.693	0.647	0.623	0.616	0.669	0.636	0.606
6 months	12	0.484	0.523	0.621	0.434	0.488	0.778	0.421	0.455
2 years	12	0.762	0.787	0.759	0.608	0.635	0.879	0.600	0.606
10 years	12	0.815	0.829	0.764	0.724	0.719	0.811	0.738	0.698

Source: Duffee (2002)

Figure 1: Summary of the estimated essentially affine $A_0(3)$ model



Affine Models and Yield Volatility

- The best model discussed above, the essentially affine model $A_0(3)$ has a major drawback: The volatility of yields is constant.
- The essentially affine model $A_1(3)$ also does relatively well in matching the properties of expected return. Moreover, it implies a time varying volatility.
- Dai and Singleton (2003) provide simulation evidence, and compare it to the two factor $A_1(2)$ and to the completely affine $A_{1R}(3)$ model.

Table 2
***ML* estimates of GARCH(1,1) parameters using historical and simulated time series of swap and Treasury yields**

GARCH(1,1)	$\overline{\sigma}$	α	β
Swap sample	0.005 (.001)	0.126 (.038)	0.657 (.062)
Model $A_{1C}(2)$	0.012 (.003)	0.102 (.040)	0.235 (.209)
Model $A_{1C}(3)$	0.008 (.000)	0.126 (.027)	0.793 (.024)
Treasury sample	0.016 (.005)	0.165 (.058)	0.749 (.069)
Model $A_{1C}(3)$	.000 (.000)	0.146 (0.075)	0.605 (.188)
Model $A_{1R}(3)$	.000 (.000)	0.164 (0.070)	NA

The GARCH model has $\sigma_t^2 = \overline{\sigma} + \alpha u_t^2 + \beta \sigma_{t-1}^2$, where u_t is the innovation from an AR(1) representation of the level of the five-year yield. Standard errors are given in parentheses.

Source: Dai and Singleton (2003)

Gaussian Linear-Quadratic Models

- There is a tension between fitting the cross-section of bond returns, the time variation in expected return, and the volatility of bond returns.
- The essentially affine model $A_1(3)$ seems to do well to match all of these conditional moments.
- A second very popular set of term structure models is specified as follows

$$\begin{array}{rrcl} \text{risk free rate} & r_t & = & \delta_0 + \boldsymbol{\delta}_1' \mathbf{X}_t + \mathbf{X}_t' \Psi \mathbf{X}_t \\ \text{risk neutral drift} & \boldsymbol{\mu}(\mathbf{X}_t) & = & \widetilde{\mathcal{K}} \left(\widetilde{\boldsymbol{\theta}} - \mathbf{X}_t \right) \\ \text{diffusion term} & \mathbf{s}(\mathbf{X}_t) & = & \boldsymbol{\Sigma} \end{array}$$

- The (RN) process for the states then is simply Gaussian

$$d\mathbf{X}_t = \widetilde{\mathcal{K}} \left(\widetilde{\boldsymbol{\theta}} - \mathbf{X}_t \right) dt + \boldsymbol{\Sigma} d\mathbf{W}_t$$

- The PDE is the same as before

$$rZ = \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial \mathbf{X}} \widetilde{\mathcal{K}} \left(\widetilde{\boldsymbol{\theta}} - \mathbf{X}_t \right) + \frac{1}{2} tr \left(\frac{\partial^2 Z}{\partial \mathbf{X}_t \partial \mathbf{X}_t'} \boldsymbol{\Sigma} \boldsymbol{\Sigma}' \right)$$

- The bond price is

$$Z(\mathbf{X}_t, t; T) = e^{A(t;T) + \mathbf{B}(t;T)' \mathbf{X}_t + \mathbf{X}_t' \mathbf{C}(t;T) \mathbf{X}_t}$$

  – where $A(t;T)$, $\mathbf{B}(t;T)$ and $\mathbf{C}(t;T)$ satisfy a set of ODEs.

Physical Measure

- What about the physical measure?
  - Since the volatility of the state variables $s(\mathbf{X}_t) = \mathbf{\Sigma}$ is constant, there is now much more flexibility in choosing the underlying market price of risk.
  - The following is a popular choice
$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1 \mathbf{X}_t$$
  - The risk natural process is
$$\begin{aligned} \mathbf{m}(\mathbf{X}_t) &= \boldsymbol{\mu}(\mathbf{X}_t) + \mathbf{s}(\mathbf{X}_t)\boldsymbol{\lambda}_t \\ &= \widetilde{\mathcal{K}}\left(\widetilde{\boldsymbol{\theta}} - \mathbf{X}_t\right) + \mathbf{\Sigma}\boldsymbol{\lambda}_0 + \mathbf{\Sigma}\boldsymbol{\lambda}_1\mathbf{X}_t \\ &= \mathcal{K}(\boldsymbol{\theta} - \mathbf{X}_t) \end{aligned}$$
- Note that even if factors $\mathbf{X}_t$ are Gaussian with constant volatility matrix $\mathbf{\Sigma}$, the model *does* imply stochastic volatility for rates and yields.
- For instance, in a one factor model: $r_t = \delta_0 + \delta_1 X_t + \delta_2 X_t^2$
- This implies
$$\begin{aligned} dr_t &= (\delta_1 + 2\delta_2 X_t)\, dX_t \\ &= (\delta_1 + 2\delta_2 X_t)\, \mathcal{K}(\theta - X_t)dt + (\delta_1 + 2\delta_2 X_t)\, \Sigma dW_t \end{aligned}$$

Estimation Issues

- If we knew the factors $\mathbf{X}_t$, estimation would be simple, as the conditional density is normal.
- However, unfortunately, there is not a one-to-one mapping between yields and factors

$$y(\mathbf{X}_t, \tau) = -\frac{A(\tau)}{\tau} - \frac{\mathbf{B}(\tau)'}{\tau}\mathbf{X}_t - \mathbf{X}_t'\frac{\mathbf{C}(t;T)}{\tau}\mathbf{X}_t$$

- Estimation is typically performed by the Simulated Method of Moments (SMM).
- Leippold and Wu (2003) suggest the use of GMM, as the normality of $\mathbf{X}$ imply analytical formulas for many moments of interest of yields $\mathbf{y}$.

The Performance of Linear Quadratic Models

- Linear quadratic models appear to be better able than essentially affine model to match important properties of the term structure of interest rates.
- Brandt and Chapman (2002) study the essentially affine model $A_0(3)$ (discussed above), $A_1(3)$ as well as the linear quadratic model QTSM(3), that is, with three factors.
- The key finding is that while all models do well in meeting unconditional premia and some predictability, the linear quadratic model does much better in matching the volatility of bond yields.
- In particular, in addition to the standard (LPY)

$$y(t+1,\tau-1) - y(t,\tau) = \phi_c + \phi_\tau \left(y(t,\tau) - y(t,1)\right)/(\tau-1) + \epsilon t + 1, \tau$$

- Brandt and Chapman (2002) also run the regression (LPV)

$$\left(y(t+1,\tau-1) - y(t,\tau) - E_t\left(y(t+1,\tau-1) - y(t,\tau)\right)\right)^2 = \mu_{0,\tau} + \sum_{i=1}^{n} \mu_{i,\tau} F_{i,t} + \epsilon(t+1,\tau)$$

- where $F_{i,t}$ is PCA level, slope and curvature factors.

The Performance of Essentially Affine Models

Table 6 (continued)

Panel B: LPY and LPV Regression Slope Coefficients.

		$\mathbb{A}_0(3)$		$\mathbb{A}_1(3)$	
	Sample Moment	Optimal W_T Fitted Moment	Diagonal W_T Fitted Moment	Optimal W_T Fitted Moment	Diagonal W_T Fitted Moment
LPY Slopes					
6-m	−0.776 (0.503)	−0.490 (0.567)	−0.004 (1.534)	−0.417 (0.713)	−0.860 (0.168)
2-y	−1.678 (0.969)	−0.892 (0.811)	−0.618 (1.094)	−2.171 (0.509)	−2.130 (0.467)
10-y	−3.832 (1.750)	−3.802 (0.017)	−3.863 (0.018)	−4.937 (0.632)	−4.038 (0.119)
LPV Slopes					
6-m on Level	0.140 (0.045)	**0.000 (3.100)**	**0.000 (3.095)**	0.069 (1.568)	0.130 (0.211)
6-m on Slope	−0.164 (0.129)	0.001 (1.275)	0.000 (1.271)	0.041 (1.582)	0.051 (1.665)
6-m on Curv.	0.198 (0.186)	−0.001 (1.068)	−0.000 (1.064)	0.005 (1.034)	0.003 (1.047)
6-m on Level	0.089 (0.032)	**0.000 (2.774)**	**0.001 (2.762)**	0.044 (1.386)	0.075 (0.439)
6-m on Slope	−0.092 (0.091)	0.001 (1.024)	0.001 (1.018)	0.038 (1.428)	0.034 (1.389)
6-m on Curv.	0.151 (0.152)	−0.001 (1.000)	−0.000 (0.995)	−0.016 (1.101)	−0.034 (1.220)
10-y on Level	0.031 (0.006)	**0.003 (4.466)**	0.033 (0.387)	0.022 (1.403)	0.031 (0.016)
10-y on Slope	0.012 (0.017)	0.002 (0.560)	0.004 (0.479)	0.017 (0.314)	0.012 (0.006)
10-y on Curv.	0.013 (0.030)	−0.004 (0.555)	0.002 (0.362)	−0.009 (0.738)	0.011 (0.057)

Source: Brandt and Chapman (2002)

The Performance of Linear Quadratic Models

Table 7 (continued)

Panel B: LPY and LPV Regression Slope Coefficients.

	Sample Moment	Std.Dev.	Optimal W_T Fitted Moment	Optimal W_T N-W t-statistic	Diagonal W_T Fitted Moment	Diagonal W_T N-W t-statistic
LPY Slope Coeff.						
6-m	−0.776	0.503	−0.598	0.353	−0.827	0.102
2-y	−1.678	0.969	−0.228	1.496	−1.505	0.179
10-y	−3.832	1.750	−1.843	1.136	−3.147	0.391
LPV Slope Coeff.						
6-m on Level	0.140	0.045	0.124	0.361	0.125	0.335
6-m on Slope	−0.164	0.129	−0.184	0.156	−0.092	0.559
6-m on Curv.	0.198	0.186	0.245	0.257	0.172	0.136
2-y on Level	0.089	0.032	0.089	0.022	0.084	0.144
2-y on Slope	−0.092	0.091	−0.108	0.184	−0.119	0.305
2-y on Curv.	0.151	0.152	0.097	0.355	0.154	0.019
10-y on Level	0.031	0.006	0.029	0.307	0.030	0.129
10-y on Slope	0.012	0.017	0.017	0.325	0.020	0.497
10-y on Curv.	0.013	0.030	−0.007	0.674	0.011	0.076

Source: Brandt and Chapman (2002)

Time Varying Risk Premia in a Habit Formation Model

- We have seen many models about the time varying risk premia.
  - However, the basic representative agent model with power utility generates a constant market price of risk.
  - Thus, the question is what type of preferences generate time varying risk premia.
  - Following Campbell and Cochrane (1999) and Menzly Santos and Veronesi (2004), Wachter (2003) and Buraschi and Jiltsov (2007) have recently proposed to use habit formation preferences as a source of time varying market price of risk in fixed income models.
  - I present the same model as in the earlier lecture notes, but with the external habit formation of Pastor and Veronesi (2005), which leads to the Quadratic Linear model described earlier.
- The representative agent maximizes

$$E\left[\int_0^\infty u\left(C_t, X_t, t\right) dt\right], \tag{7}$$

- where the instantaneous utility function is give by

$$u\left(C_t, X_t, t\right) = \begin{cases} e^{-\rho t} \frac{\left(C_t - X_t\right)^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 1 \\ e^{-\rho t} \log\left(C_t - X_t\right) & \text{if } \gamma = 1 \end{cases} \tag{8}$$

- X_t is an external habit level, as in Campbell and Cochrane (1999).

The Surplus Consumption Ratio Dynamics

- Consider now the following quantity

$$S_t = \frac{C_t - X_t}{C_t} \tag{9}$$

- Campbell and Cochrane (1999) call S_t the *Surplus Consumption Ratio*.
- This is a key quantity in determining the properties of the market price of risk:

$$\pi_t = e^{-\rho t} \frac{\partial u\left(C_t, X_t\right)}{\partial C_t} = e^{-\rho t}\left(C_t - X_t\right)^{-\gamma} = e^{-\rho t} C_t^{-\gamma} S_t^{-\gamma}$$

- The surplus consumption ratio acts as a "preference shock", as it changes the curvature of the utility function: γS_t^{-1}.
- Clearly, we must have $S_t \in [0, 1]$
  - Problem: this cannot be ensured in endowment economies when X_t is an average of past consumption.
- In addition, from first principles, S_t is:
  - Mean reverting: This is a consequence of habit formation and the fact that X_t is slow moving.
  - Perfectly correlated with innovations to consumption growth.
  - The volatility of surplus is time varying.

Campbell and Cochrane Solution

- Campbell and Cochrane (1999) had a great intuition:
  - Specify the mean reverting dynamics for *log* surplus $s_t = \log(S_t)$
  - Specify the log surplus volatility $\lambda(s_t)$ in a way to ensure $S_t = exp(s_t) \in [0,1]$.
- In addition, they specified $\lambda(s_t)$ to obtain specific properties of the interest rate process r_t
  - Unfortunately, their specification does not yield closed form solutions for prices.
- Pastor and Veronesi (2005) simply use a different transformation of surplus consumption ratio and obtain closed form formulas.

$$\begin{aligned} S_t &= e^{s_t} \\ s_t &= a_0 + a_1 y_t + a_2 y_t^2 \\ dy_t &= k_y \left(\overline{y} - y_t\right) dt + \sigma_y dW_{c,t} \end{aligned}$$

- Choosing a_i appropriately (in particular, $a_2 < 0$) $\Longrightarrow s_t < 0 \Longrightarrow S_t \in [0,1]$.
- In addition, we must have $\partial s(y)/\partial y > 0$, so that positive shocks to consumption $dW_{c,t}$ translate in positive shocks to the surplus consumption ratio S_t.
  - We need to have $a_1 + 2a_2 y_t > 0$. This restriction can be enforced with high probability, as y_t is normally distributed, and we have many free parameters.

GDP growth and Inflation

- The rest of the model can be assumed as in the earlier lecture (with power utility).
- In particular, let $c_t = log(C_t)$ and $q_t = log(Q_t)$ be log consumption and log CPI. Then

$$\begin{aligned} dc_t &= g_t dt + \sigma_c dW_{c,t} \\ dq_t &= i_t dt + \sigma_q dW_{i,t} \end{aligned}$$

- Assume that $X_t = (g_t, i_t, y_t)'$ follows the process the process

$$d\mathbf{X}_t = \mathbf{K}\left(\mathbf{\Theta} - \mathbf{X}_t\right) dt + \mathbf{\Sigma} d\mathbf{W}_t$$

- Parameters must be chosen to make sure that g_t and i_t do not depend on y_t, as it would make little economic sense.
- The SDF is given by

$$\pi_t = e^{-\eta t - \gamma\left(c_t + a_0 + a_1 y_t + a_2 y_t^2\right) - q_t}$$

- This gives the dynamics

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \sigma_\pi dW_{c,t} - \sigma_q dW_{q,t}$$

The Risk Free Rate and the Market Price of Risk

- The interest rate has a linear quadratic structure

$$r_t = \delta_0 + \gamma g_t + i_t + \delta_y y_t + \delta_{yy} y_t^2$$

- where δ_0, δ_y and δ_{yy} are given in the appendix below.

- The market price of risk has also a linear structure:

$$\sigma_\pi = \lambda_t = \gamma \left(\sigma_c + \sigma_y(a_1 + 2a_2 y_t)\right)$$

- $\Longrightarrow$ The bond pricing formula must be the same as the one obtained earlier, with factors GDP/ consumption growth g_t, expected inflation i_t and habit y_t.

- This model has never been formally studied as a model of interest rates, as Pastor Veronesi (2005) simply use it in a model of IPOs
  - It may be interesting to calibrate the model to reasonable consumption based parameters and see the implications for the term structure.

Conclusion

1. Term structure models have gone through an exciting period of innovation in the last few years.
2. Researchers have become more and more interested in explaining the dynamics of interest rates, rather than obtaining a nice "fit"
   - The inability of standard affine models of explaining expected returns had frustrated many researchers.
3. The key breakthrough (almost obvious in retrospect) was to change the specification of the market price of risk
   - Care needs to be applied here, as the market price of risk must be specified in a way to preserve no-arbitrage opportunities. There are some technical restrictions that need to be satisfied.
4. Additional models also boil down to a specification of market price of risk that is not too tightly linked to the dynamics of interest rates.
   - E.g. Gaussian Linear Quadratic Models, Regime shifts and non-affine habit formation (Buraschi and Jiltsov (2007)).
5. Yet, in all of these models, the factors are latent and are always estimated from yields. The next step is to put back some macro-economics in the fixed income literature.

Appendix

The constants in the interest rate model are

$$
\begin{aligned}
\delta_0 &= \eta + \gamma a_1 k \overline{y} - \frac{1}{2}\left(\gamma^2\sigma_c^2 + \gamma^2 a_1^2 \sigma_y^2 + \sigma_q^2\right) - a_1\gamma^2\sigma_y\sigma_c - a_1\gamma\sigma_y\sigma_q\rho_{cq} - \gamma\sigma_c\sigma_q\rho_{c,q} + \gamma a_2 \sigma_y^2 \\
\delta_y &= \left(2\gamma k\overline{y}a_2 - \gamma a_1 k - \gamma^2 a_1 a_2 \sigma_y^2 - 2a_2\gamma^2\sigma_y\sigma_c - 2\gamma\sigma_y\sigma_q\rho_{cq}a_2\right) \\
\delta_{yy} &= -2\gamma k a_2 - 2\gamma^2 a_2^2 \sigma_y^2
\end{aligned}
$$