Modern Dynamic Asset Pricing Models

Lecture Notes 1.

Dynamic Portfolio Allocation Strategies

Pietro Veronesi

Graduate School of Business,
University of Chicago
CEPR, NBER

Outline

1. Review of Merton / Samuelson Portfolio Allocation Problem
   - The Puzzles
2. Strategic Asset Allocation under Predictability of Stock Returns
   - The Problem and its solution
   - Implications for Dynamic Asset Allocation
3. Learning about Average Returns
   - Implications for Dynamic Asset Allocation
   - Comparison with the case of Predictability
4. Strategic Asset Allocation with Model Misspecification
   - The Problem and Its solution
   - The Example of Constant Investment Opportunity Set
5. Conclusion

Review of Merton/Samuelson Portfolio Allocation Problem

- There are n stocks. Stock i return

$$dR_t^i = \frac{dS_t^i + D_t^i dt}{S_t^i}$$

- $d\mathbf{R}_t = (dR_t^1, .., dR_t^n)'$
- Assume:

$$d\mathbf{R}_t = \boldsymbol{\mu} dt + \boldsymbol{\sigma} d\mathbf{B}_t$$

- $d\mathbf{B}_t = (dB_t^1, ..., dB_t^n)$ = vector of *independent* Brownian motions.
- **Investor problem**:

$$J(W_0, 0) = \max_{\{(C_t),(\boldsymbol{\theta}_t)\}} E_0 \left[\int_0^T u(C_t, t)\, dt \right]$$

- subject to

$$dW_t = \left\{ W_t \left(\boldsymbol{\theta}_t'(\boldsymbol{\mu} - r\mathbf{1}_n) + r \right) - C_t \right\} dt + W_t \boldsymbol{\theta}_t' \boldsymbol{\sigma} d\mathbf{B}_t$$

The Bellman Equation

- Bellman Equation:

$$0 = \sup_{(C_t,\boldsymbol{\theta})} u(C,t) + E\left[dJ(W,t)\right]/dt$$

- with boundary condition $J(W_T, T) = 0$
- Why this form?
  - The discrete time Bellman equation over a small Δ

$$J(W_t,t) = \max_{C,\theta} \left\{ u(C,t)\Delta + E\left[J(W_{t+\Delta}, t+\Delta) | W_t\right] \right\}$$

$$\Longrightarrow 0 = \max_{C,\theta} u(c,t)\Delta + E_t\left[J(W_{t+\Delta}, t+\Delta) - J(W_t,t)\right]$$

- Note that by Ito's Lemma:

$$\begin{aligned} E[dJ(W,t)]/dt &= J_t + J_W E_t[dW]/dt + \frac{1}{2} J_{WW} E_t[dW^2]/dt \\ &= J_t + J_W \left\{ W_t \left(\boldsymbol{\theta}_t'(\mu - r) + r \right) - C_t \right\} + \frac{1}{2} J_{WW} W_t^2 \boldsymbol{\theta}_t' \boldsymbol{\sigma}\boldsymbol{\sigma}' \boldsymbol{\theta}_t \end{aligned}$$

The Optimal Consumption and Portfolio Allocation

- FOC with respect to C:

$$u_c\left(C_t, t\right) = J_W\left(W, t\right)$$

  - Example: Power utility

$$u\left(C_t, t\right) = e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma} \quad \Longrightarrow \quad C_t = e^{-\frac{\rho}{\gamma}t} J_W\left(W, t\right)^{-\frac{1}{\gamma}}$$

- FOC with respect to $\boldsymbol{\theta}_t$:

$$\boldsymbol{\theta}_t = \frac{1}{RRA(W)}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\left(\boldsymbol{\mu} - r\mathbf{1}_n\right)$$

  - where

$$RRA(W) = -\frac{W J_{WW}\left(W, t\right)}{J_W\left(W, t\right)}$$

- We now solve for $J(W, t)$ in the power utility case.

The Explicit Solution via an Ordinary Differential Equation

1. Conjecture:

$$J(W,t) = e^{-\rho t}\frac{W^{1-\gamma}}{1-\gamma}F(t)^{\gamma}$$

2. Compute J_t, J_W and J_{WW};
3. Optimal consumption and portfolio holdings:

$$C_t = WF(t)^{-1}; \quad \text{and} \quad \boldsymbol{\theta}_t = \frac{1}{\gamma}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\left(\boldsymbol{\mu} - r\mathbf{1}\right)$$

4. To find $F(t)$, substitute J_t, J_W and J_{WW} and optimal strategies in Bellman equation

$$0 = e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma} - \rho J + J\gamma\frac{F_t}{F} + W^{-1}(1-\gamma)J\left(W_t\left(\boldsymbol{\theta}_t'\left(\boldsymbol{\mu} - r\mathbf{1}_n\right) + r\right) - C_t\right)$$

$$-\frac{1}{2}\gamma(1-\gamma)W^{-2}JW_t^2\boldsymbol{\theta}_t'\boldsymbol{\sigma}\boldsymbol{\sigma}'\boldsymbol{\theta}_t$$

The Explicit Solution via a Ordinary Differential Equation

5. Simplify all that can be simplified, to find the ODE

$$0 = 1 - aF(t) + F_t$$

where $F(T) = 0$ and

$$a = \frac{1}{\gamma}\left\{\rho - (1-\gamma)\,r - \frac{1-\gamma}{2\gamma}\left(\boldsymbol{\mu} - r\mathbf{1}_n\right)'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\left(\boldsymbol{\mu} - r\mathbf{1}_n\right)\right\}$$

6. The solution is

$$F(t) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

- As $t \to T$, consume a higher fraction of wealth.

7. The last point is to verify that "Conjecture" is indeed optimal.

The Puzzles

- For $n = 1$

$$\theta_t = (\mu - r) / (\gamma \sigma^2)$$

1. θ_t is independent of age t, and thus of remaining life $T - t$.
   - Against empirical evidence: an inverted U shaped θ_t
   - Against the typical recommendation of portfolio advisors.
2. Too large θ. Using $\mu - r = 7\%$ and $\sigma = 16\%$

Table: Portfolio Allocation

	Risk Aversion γ				
	2	4	6	8	10
θ	136%	68%	45%	34 %	27 %

   - Typical household holds between 6 % to 20 % in equity.
   - Conditional on participation, $\approx 40\%$ of financial assets.

Strategic Asset Allocation with Time Varying Expected Returns

- n stocks:

$$d\mathbf{R}_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma} d\mathbf{B}_t$$

  - $\boldsymbol{\mu}_\mathbf{t} = E_t\left[d\mathbf{R}_t\right]$ is now time varying.

- For convenience (later), denote the expected *excess* return

$$\boldsymbol{\lambda}_\mathbf{t} = \boldsymbol{\mu}_t - r\mathbf{1}_n$$

- Assume a VAR process

$$d\boldsymbol{\lambda}_t = \left(\mathbf{A}_0 + \mathbf{A}_\mathbf{1}\boldsymbol{\lambda}_t\right) dt + \boldsymbol{\Sigma} d\mathbf{B}_t$$

- Note:
  - Assume $d\mathbf{B}_t$ is now $n \times m$.
  - E.g. $n = 1$ (1 stock), $m = 2$ (two shocks) with

$$\boldsymbol{\sigma} = (\sigma_1, 0) \quad \boldsymbol{\Sigma} = (\Sigma_1, \Sigma_2) \quad \Longrightarrow \text{Cov}\left(dR, d\lambda\right) = \boldsymbol{\sigma}\boldsymbol{\Sigma}' = \sigma_1\Sigma_1$$

The Bellman Equation with Time Varying Expected Returns

- **Investor problem**:

$$J(W_0, \lambda_0, 0) = \max_{\{(C_t),(\boldsymbol{\theta}_t)\}} E_0\left[\int_0^T u(C_t, t)\, dt\right]$$

- subject to

$$dW_t = \left\{W_t\left(\boldsymbol{\theta}_t'\boldsymbol{\lambda}_t + r\right) - C_t\right\}dt + W_t\boldsymbol{\theta}_t'\boldsymbol{\sigma}\, d\mathbf{B}_t$$

- The Bellman equation is

$$0 = \max_{C_t,\theta_t} e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma} + E_t\left[dJ_t\right]/dt$$

- with

$$E_t\left[dJ_t\right]/dt = J_t + J_W E_t\left[dW_t\right] + \frac{1}{2}J_{WW}E_t\left[dW_t^2\right]$$

$$+\mathbf{J}_\lambda' E_t\left[d\boldsymbol{\lambda}_t\right] + \mathbf{J}_{W\lambda}E_t\left[d\boldsymbol{\lambda}_t dW_t\right] + \frac{1}{2}tr\left(\mathbf{J}_{\lambda\lambda}E\left[d\boldsymbol{\lambda}_t d\boldsymbol{\lambda}_t'\right]\right)$$

Optimal Consumption and Portfolio Allocation

- Substitute expectations in Bellman equation:

$$0 = \max_{C_t, \boldsymbol{\theta}_t} e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} + J_t + J_W \left(W_t \left(\boldsymbol{\theta}_t' \boldsymbol{\lambda}_t + r \right) - C_t \right) + \frac{1}{2} J_{WW} W_t^2 \boldsymbol{\theta}_t' \boldsymbol{\sigma} \boldsymbol{\sigma}' \boldsymbol{\theta}_t$$
$$+ \mathbf{J}_\lambda' \left(\mathbf{A}_0 + \mathbf{A}_1 \boldsymbol{\lambda}_t \right) + \mathbf{J}_{W\lambda} W_t \boldsymbol{\Sigma} \boldsymbol{\sigma}' \boldsymbol{\theta}_t + \frac{1}{2} tr \left(\mathbf{J}_{\lambda\lambda} \boldsymbol{\Sigma} \boldsymbol{\Sigma}' \right)$$

- FOC with respect to C_t:

$$C_t = e^{-\frac{\rho}{\gamma} t} J_W^{-\frac{1}{\gamma}}$$

  – Same form as before.
  – But recall that J_W is not different.

- FOC with respect to $\boldsymbol{\theta}_t$:

$$\boldsymbol{\theta}_t = \frac{1}{RRA(W_t)} \left(\boldsymbol{\sigma} \boldsymbol{\sigma}' \right)^{-1} \boldsymbol{\lambda}_t - \left(\boldsymbol{\sigma} \boldsymbol{\sigma}' \right)^{-1} \boldsymbol{\sigma} \boldsymbol{\Sigma}' \frac{\mathbf{J}_{W\lambda}}{J_{WW} W}$$

  – There is one additional term.

Optimal Portfolio Allocation

- Optimal Portfolio Allocation:

$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_t^M + \boldsymbol{\theta}_t^H$$

- Myopic Demand

$$\boldsymbol{\theta}_t^M = \frac{1}{RRA(W_t)} \left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1} \boldsymbol{\lambda}_t$$

  – Same as before.

- Hedging Demand

$$\boldsymbol{\theta}_t^H = -\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1} \boldsymbol{\sigma}\boldsymbol{\Sigma}' \frac{\mathbf{J}_{W\lambda}}{J_{WW}W}$$

  – Recall that expected returns $\boldsymbol{\lambda}_t$ also (obviously) affect intertemporal utility.

  – $\Longrightarrow$ The asset allocation must "hedge" against the negative impact that the variation in expected returns has on the marginal utility.

- If $\boldsymbol{\theta}_t^H$ depends on age (t) and is negative, we may "resolve" the two puzzles.

Optimal Portfolio Allocation under Power Utility

- Solving this problem is substantially more complicated.
- Conjecture 1:

$$J\left(W_t, \boldsymbol{\lambda}_t, t\right) = e^{-\rho t}\frac{W_t^{1-\gamma}}{1-\gamma}F\left(\boldsymbol{\lambda}_t, t\right)^{\gamma}$$

- Compute J_t, J_W, J_{WW} , $\mathbf{J}_{W\boldsymbol{\lambda}}$, $\mathbf{J}_\lambda$ and $\mathbf{J}_{\lambda\lambda}$.
- This yields

$$C_t = W_t F^{-1} \quad \text{and} \quad \boldsymbol{\theta}_t = \frac{1}{\gamma}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\lambda}_t + \left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\sigma}\boldsymbol{\Sigma}'\frac{\mathbf{F}_\lambda}{F}$$

- To solve for $F\left(\lambda, t\right)$, substitute everything into the Bellman equation.

The Bellman Equation and its Solution

$$
\begin{aligned}
0 \;=\; & F^{-1} + ((1-\gamma)\, r - \rho)\,\frac{1}{\gamma} + \frac{F_t}{F} + \frac{1}{2} tr\left(\frac{\mathbf{F}_{\lambda\lambda}}{F}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right) + \frac{(1-\gamma)}{2\gamma^2}\boldsymbol{\lambda}_t'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\lambda}_t + \\
& + \frac{(1-\gamma)}{\gamma}\frac{\mathbf{F}_\lambda'}{F}\boldsymbol{\Sigma}\boldsymbol{\sigma}'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\lambda}_t + \frac{\mathbf{F}_\lambda'}{F}\left(\mathbf{A}_0 + \mathbf{A}_1\boldsymbol{\lambda}_t\right) + \\
& + \frac{1}{2}(1-\gamma)\, tr\left(\left(\frac{\mathbf{F}_\lambda}{F}\frac{\mathbf{F}_\lambda'}{F}\right)\left(\boldsymbol{\Sigma}\boldsymbol{\sigma}'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\sigma}\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)\right)
\end{aligned}
$$

- This is horrible. There is:
  - A quadratic term in $\boldsymbol{\lambda}_t$;
  - A linear term in $\boldsymbol{\lambda}_t$;
  - A quadratic term in $\mathbf{F}_\lambda$.

- Yet, by applying recent techniques developed in Fixed Income, an analytical solution exists for the case

$$
\boldsymbol{\Sigma}\boldsymbol{\sigma}'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\sigma}\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}\boldsymbol{\Sigma}' = \mathbf{0}
$$

Towards an Analytical Solution

- Conjecture 2:

$$F(\boldsymbol{\lambda},t;T) = \int_t^T f(\boldsymbol{\lambda},t;\tau)\, d\tau$$

- with $f(\lambda, t, t) = 1$.
- After some algebra, we find the following PDE for $f(\boldsymbol{\lambda}_t, t; \tau)$:

$$0 = ((1-\gamma)r - \rho)\frac{1}{\gamma}f + f_t + \frac{1}{2}tr\left(\mathbf{f}_{\boldsymbol{\lambda}\boldsymbol{\lambda}}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right) + \frac{(1-\gamma)}{2\gamma^2}\boldsymbol{\lambda}_t'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\lambda}_t f +$$
$$+\frac{(1-\gamma)}{\gamma}\mathbf{f}_{\boldsymbol{\lambda}}'\boldsymbol{\Sigma}\boldsymbol{\sigma}'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\lambda}_t + \mathbf{f}_{\boldsymbol{\lambda}}'\left(\mathbf{A}_0 + \mathbf{A}_1\boldsymbol{\lambda}_t\right)$$

- Perhaps this does not look any better to most, but it is a very standard PDE in Fixed Income Asset Pricing.
  - The solution is an exponential linear-quadratic function of $\boldsymbol{\lambda}_t$

An Analytical Solution

- Use method of undetermined coefficients.
- Conjecture 3:

$$f(\boldsymbol{\lambda}, t; \tau) = e^{\alpha_0(t;\tau) + \boldsymbol{\alpha}_1(t;\tau)'\boldsymbol{\lambda}_t + \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\alpha}_2(t;\tau)\boldsymbol{\lambda}_t}$$

1. Take the derivatives f_t, $\mathbf{f}_\lambda$ and $\mathbf{f}_{\lambda\lambda}$
2. Substitute and pool terms together

- to obtain

$$\begin{aligned} 0 &= ((1-\gamma) r - \rho)\frac{1}{\gamma} + \frac{\partial \alpha_0(t;\tau)}{\partial t} + \boldsymbol{\alpha}_1(t,\tau)'\mathbf{A}_0 + \frac{1}{2} tr(\boldsymbol{\alpha}_2(t,\tau)\boldsymbol{\Sigma}\boldsymbol{\Sigma}') + \frac{1}{2} tr\left(\boldsymbol{\alpha}_1(t,\tau)\boldsymbol{\alpha}_1(t,\tau)'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right) \\ &+ \left(\frac{\partial \boldsymbol{\alpha}_1(t,\tau)'}{\partial t} + (1-\gamma)\boldsymbol{\alpha}_1(t,\tau)'\boldsymbol{\Sigma}\boldsymbol{\sigma}'\frac{1}{\gamma}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1} + \boldsymbol{\alpha}_1(t,\tau)'\mathbf{A}_1 + \mathbf{A}_0'\boldsymbol{\alpha}_2(t,\tau) + \boldsymbol{\alpha}_1(t,\tau)\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\alpha}_2(t,\tau)\right)\boldsymbol{\lambda}_t \\ &+ tr\left(\left(\frac{1}{2}\frac{\partial \boldsymbol{\alpha}_2(t,\tau)}{\partial t} + \frac{1}{2}(1-\gamma)\frac{1}{\gamma^2}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1} + (1-\gamma)\boldsymbol{\alpha}_2(t,\tau)'\boldsymbol{\Sigma}\boldsymbol{\sigma}'\frac{1}{\gamma}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1} + \boldsymbol{\alpha}_2(t,\tau)'\mathbf{A}_1 + \frac{1}{2}\boldsymbol{\alpha}_2(t,\tau)'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\alpha}_2(t,\tau)\right)\boldsymbol{\lambda}_t\boldsymbol{\lambda}_t'\right) \end{aligned}$$

- In order for the right hand side to be zero independently of $\boldsymbol{\lambda}_t$, the following must hold.

An Analytical Solution

- A system of ODE:

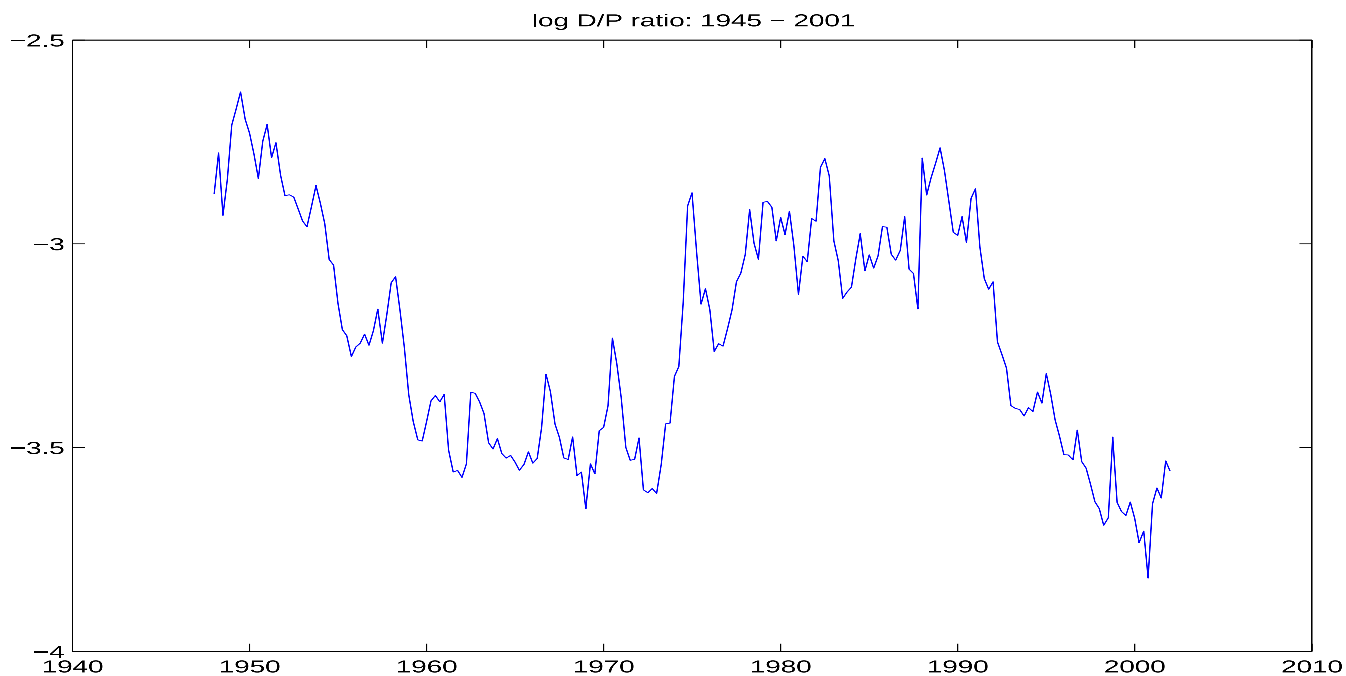
$$
\begin{aligned}
0 &= \frac{\partial \boldsymbol{\alpha}_2(t,\tau)}{\partial t} + (1-\gamma)\frac{1}{\gamma}\left(\frac{1}{\gamma} + 2\boldsymbol{\alpha}_2(t,\tau)'\boldsymbol{\Sigma}\boldsymbol{\sigma}'\right)(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1} + 2\boldsymbol{\alpha}_2(t,\tau)'\mathbf{A}_1 + \boldsymbol{\alpha}_2(t,\tau)'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\alpha}_2(t,\tau) \\
0 &= \frac{\partial \boldsymbol{\alpha}_1(t,\tau)'}{\partial t} + (1-\gamma)\boldsymbol{\alpha}_1(t,\tau)'\boldsymbol{\Sigma}\boldsymbol{\sigma}'\frac{1}{\gamma}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1} + \boldsymbol{\alpha}_1(t,\tau)'\mathbf{A}_1 + \mathbf{A}_0'\boldsymbol{\alpha}_2(t,\tau) + \boldsymbol{\alpha}_1(t,\tau)\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\alpha}_2(t,\tau) \\
0 &= \frac{\partial \alpha_0(t;\tau)}{\partial t} + ((1-\gamma)r - \rho)\frac{1}{\gamma} + \boldsymbol{\alpha}_1(t,\tau)'\mathbf{A}_0 + \frac{1}{2}tr(\boldsymbol{\alpha}_2(t,\tau)\boldsymbol{\Sigma}\boldsymbol{\Sigma}') + \frac{1}{2}tr\left(\boldsymbol{\alpha}_1(t,\tau)\boldsymbol{\alpha}_1(t,\tau)'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)
\end{aligned}
$$

- with final conditions $\boldsymbol{\alpha}_i(\tau,\tau) = 0$, $i = 0, 1, 2$.
- These ODEs can be easily solved numerically, independently of the dimension.
  - Just start with the final condition at τ and move backwards over time (it is three lines of code: one for each ODE).

Application 1: Portfolio Allocation under Predictability

- Let $n = 1$ and dR_t be the return on the aggregate stock market.
- Much of the literature uses the log dividend price ratio as a predictor.
- Let $x_t = \log(D_t/P_t)$ and let it follow the mean reverting process

$$dx_t = (\eta - \phi x_t)\, dt + \Sigma_{x1} dB_t^1$$



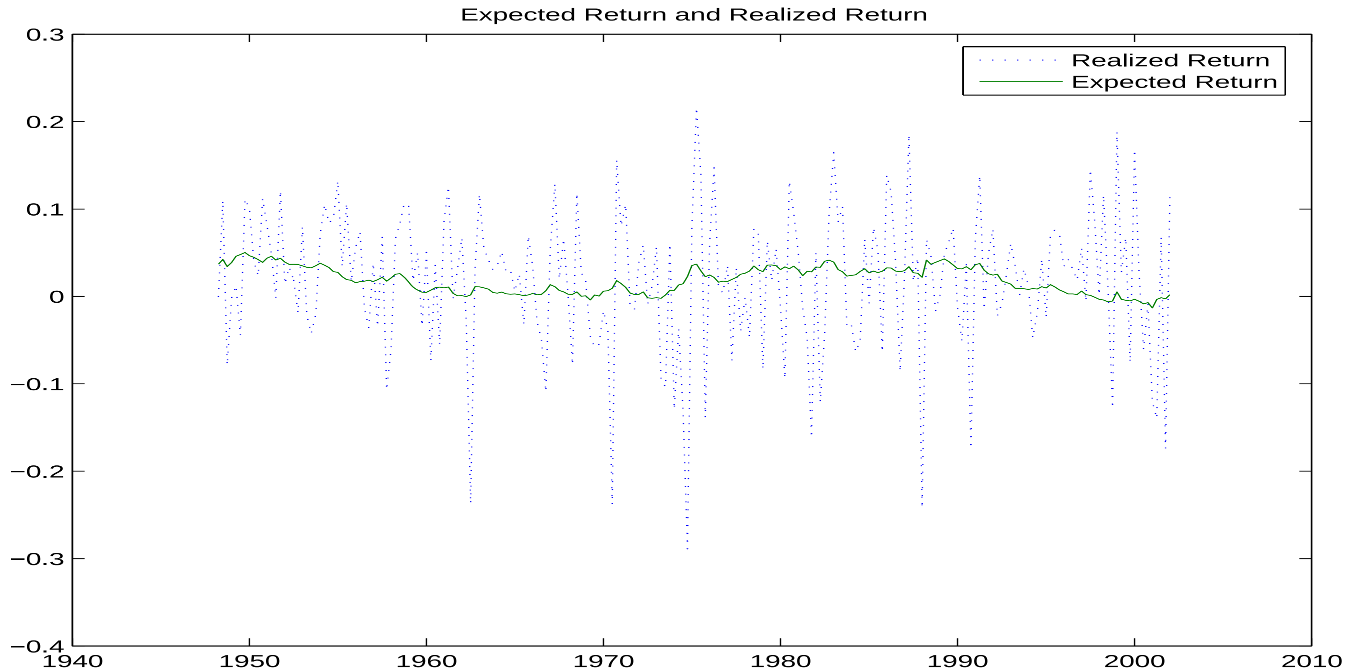
Application 1: Portfolio Allocation under Predictability

- Using x_t a predictor of excess stock returns, we can estimate

$$R_{t,t+dt} = \tilde{\beta}_0 + \tilde{\beta}_1 x_t + \epsilon_{t+dt}$$

Sample: 1947 - 2001. $dt = .25$

$\tilde{\beta}_0$	(t-stat)	$\tilde{\beta}_1$	(t-stat)	R^2
0.1898	(3.7042)	0.0531	(3.2424)	3.53%



Application 1: Portfolio Allocation under Predictability

- The annualized expected return $\lambda_t = E_t[R_{t,t+dt}/dt]$ is given by

$$\lambda_t = \beta_0 + \beta_1 x_t$$

- with $\beta_i = \tilde{\beta}_i/dt$
- Ito's Lemma implies

$$d\lambda_t = (A_0 + A_1\lambda_t)\,dt + \Sigma_1 dB_t^1$$

with

$$A_0 = \beta_1\eta + \phi\beta_0; A_1 = -\phi; \Sigma_1 = \beta_1\Sigma_{x1}$$

- The process for stock returns is

$$dR_t = (r + \lambda_t)\,dt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2$$

Application 1: Portfolio Allocation under Predictability

Model:

$$
\begin{aligned}
d\lambda_t &= (A_0 + A_1\lambda_t)\,dt + \Sigma_1 dB_t^1 \\
dR_t &= (r + \lambda_t)\,dt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2
\end{aligned}
$$

Sample: 1947 - 2001. $dt = .25$

A_0	A_1	Σ_1	σ_1	σ_2
0.0077	-0.1405	-0.0317	0.1183	0.1057

- **Note 1**: Negative Σ_1 simply means $Cov(dR, d\lambda) = \Sigma_1\sigma_1 = -.0038 < 0$
  - Positive shocks to dividend yield increase expected returns but are *contemporaneously* negatively correlated with returns.
    - This is intuitive: dividend yield moves mainly because of prices.
    - If $P_t \downarrow \Longrightarrow dR_t < 0$ and $log(D/P) \uparrow$

A bad news ($dR < 0$) is not very bad, as it increases expected returns

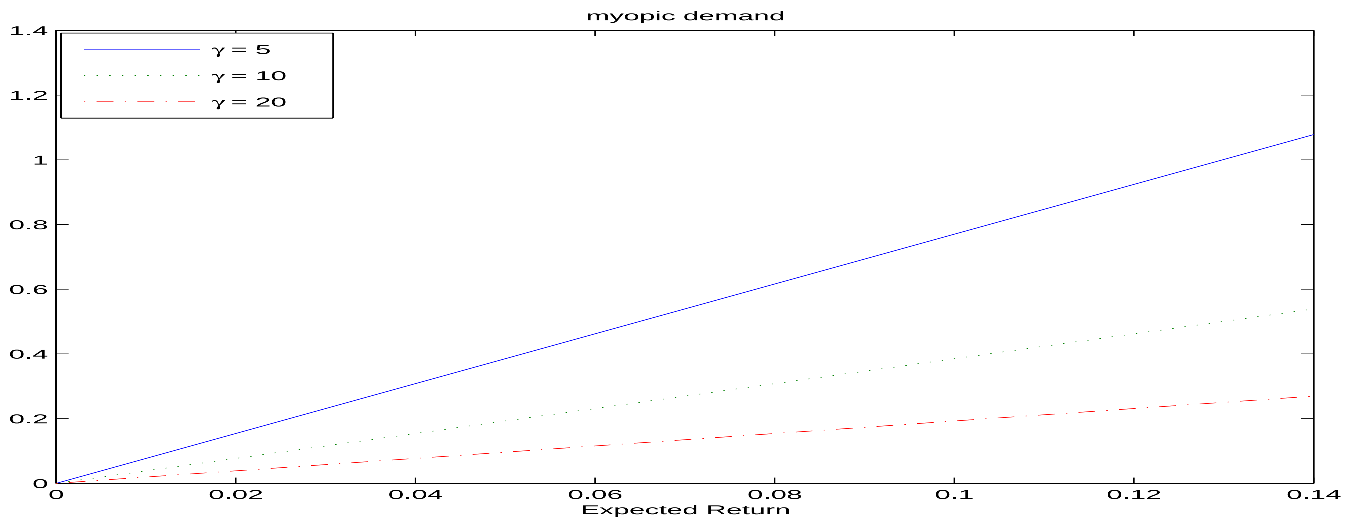
Application 1: Portfolio Allocation under Predictability

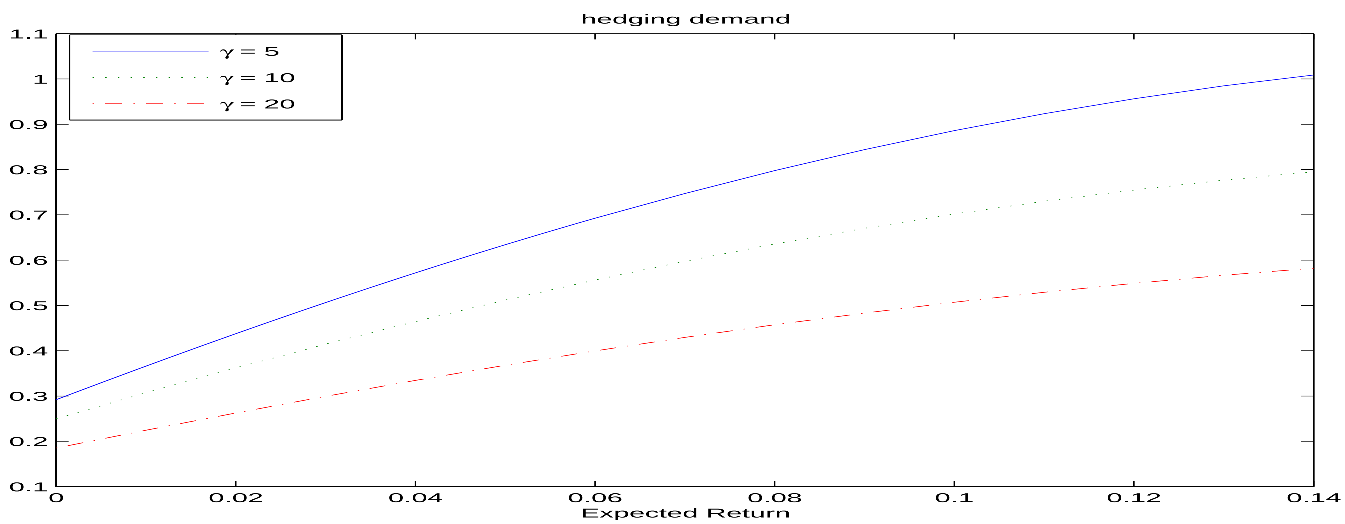
- **Note 2**: The condition for an exact analytical solution is violated:

$$\Sigma\sigma'\left(\sigma\sigma'\right)^{-1}\sigma\Sigma-\Sigma\Sigma'=0\Rightarrow\frac{\left(\sigma_1\Sigma_1\right)^2}{\sigma_1^2+\sigma_2^2}=\Sigma_1^2\Rightarrow\sigma_2^2=0$$

- $\Longrightarrow$ Exact formula really holds under the assumption of complete markets.
  - Stock returns span all of the uncertainty.
- Instead, we found $\sigma_2>0$.
  - Part of the problem is the use of quarterly data. At monthly frequency the (negative) correlation between returns and dividend yield is higher.
  - For the sake of argument, I will assume a perfect negative correlation between returns and dividend yield.
    - $*$ In what follows I then use $\sigma_1=.1612$ and $\sigma_2=0$.

Myopic and Hedging Demand for Various Risk Aversion Parameter

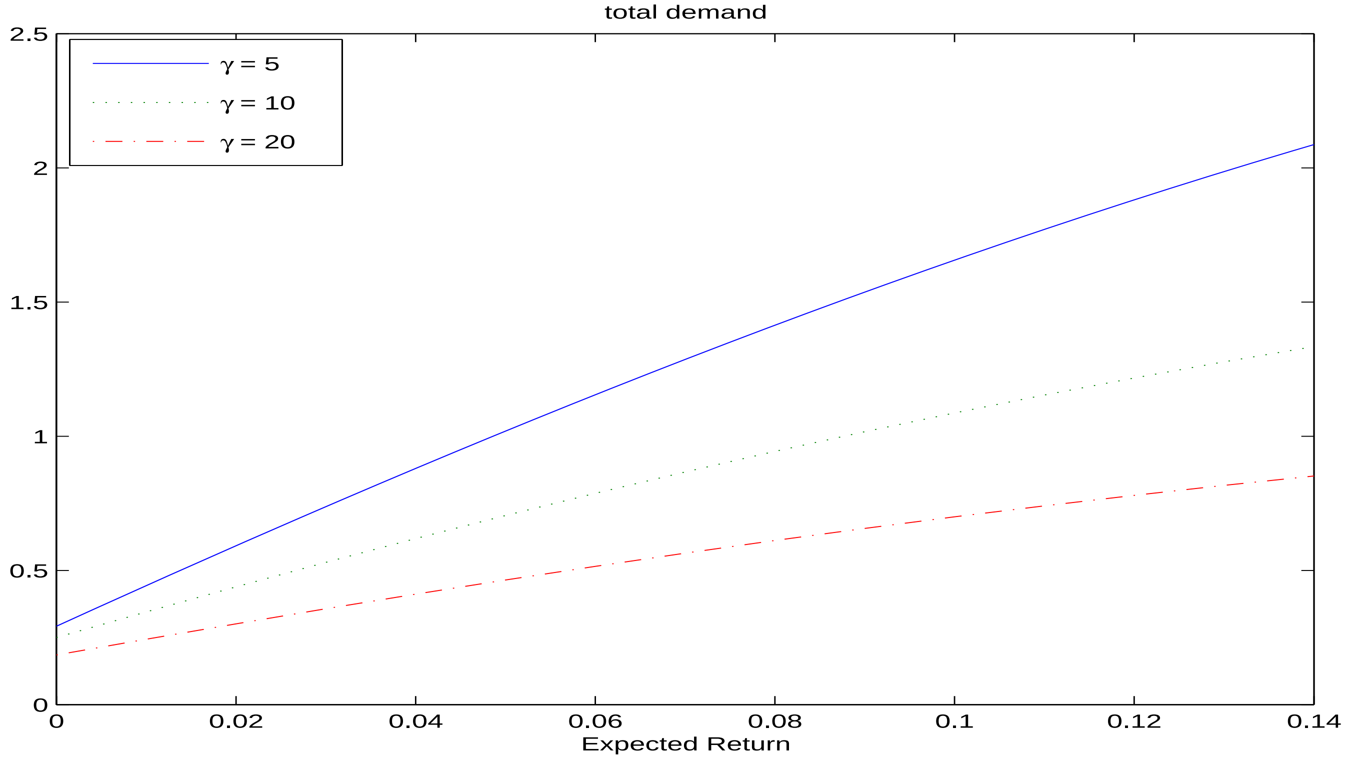




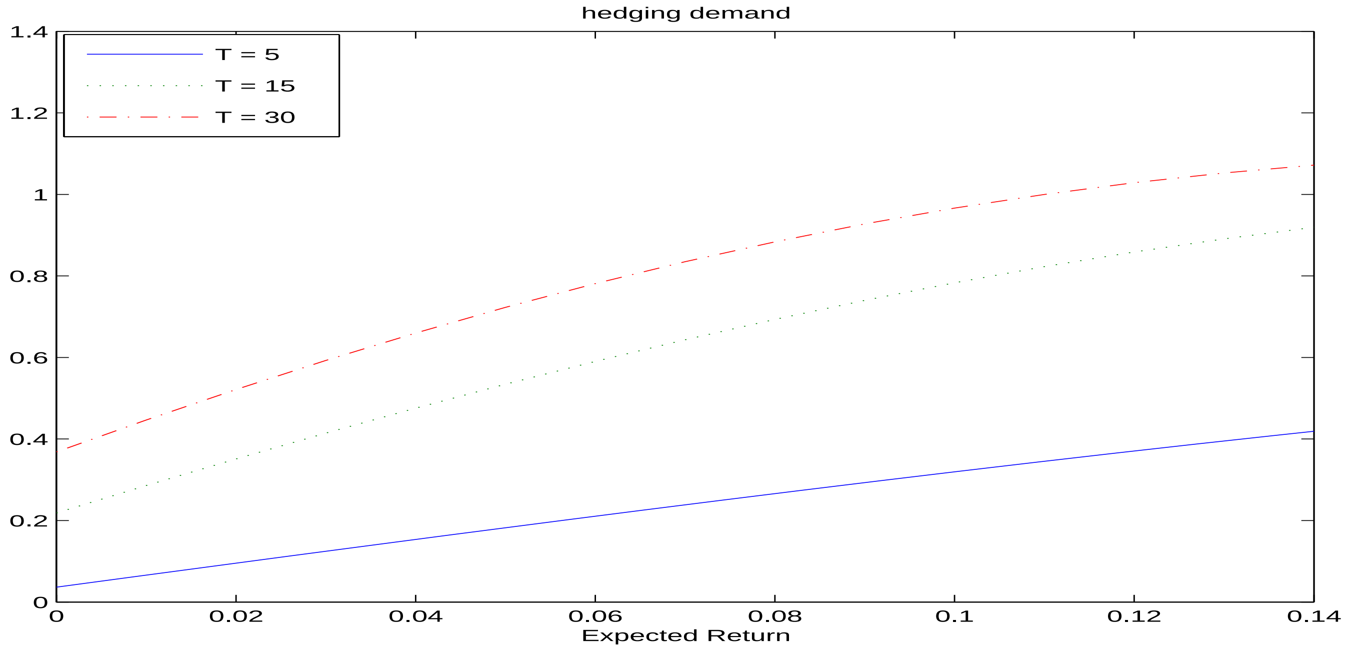
Hedging Demand with Predictable Returns

- **Finding 1:** The hedging demand is positive.
- The intuition is simple:
  - If we have a bad shock to returns, we have that μ_t increases (intuitively, the D/P increases, implying higher expected return).
  - But a higher λ_t implies that investor now want to buy more of the stock.
  - Anticipating this correlation, the investor buys more of the stock today, compared to the case where the hedging demand is zero.
- This finding is bad news for the portfolio holding puzzle:
  - We already showed that the agent would hold too much of the stock even with simple myopic demand (no time varying investment opportunity set).
    - The total demand now of the stock is even higher, deepening the puzzle.

Total Demand with Predictable Returns

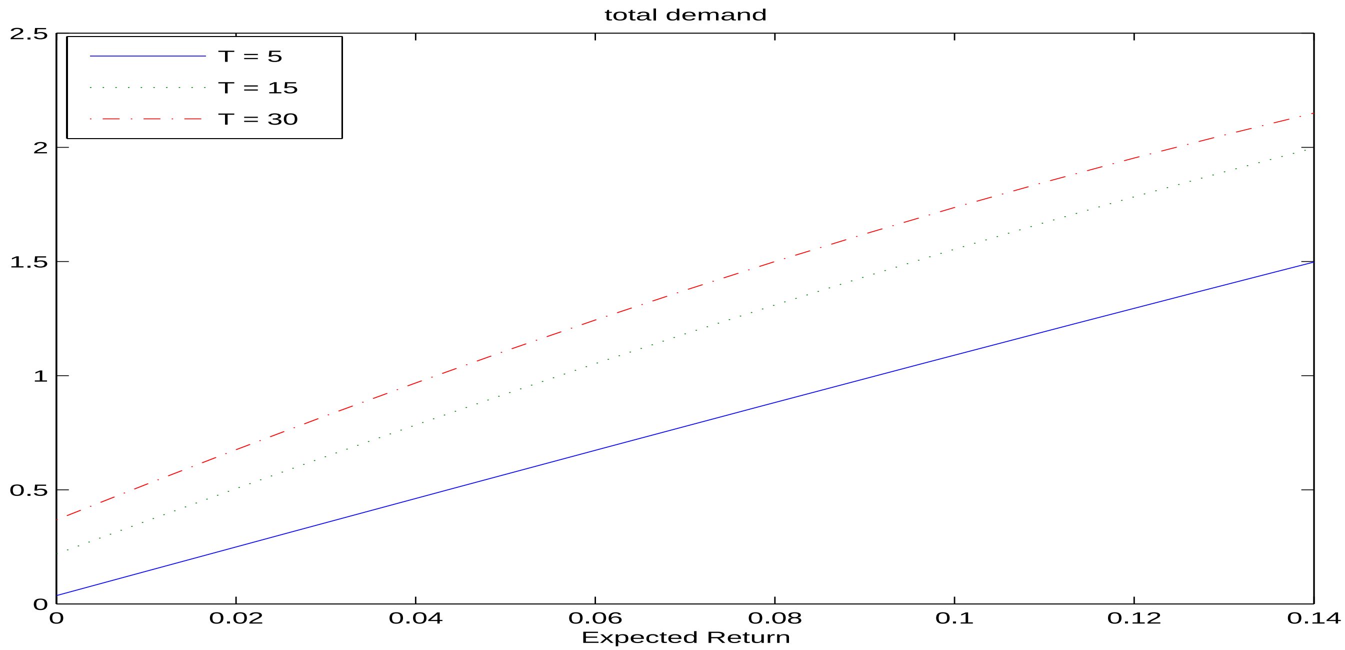


Hedging Demand for Various Life Expectancy



- **Finding 2:** Hedging demands help to address the life - cycle allocation puzzle.
  - As it can be see, the shorter the life expectancy T the lower the share in stocks, especially if current expected return is high.
  - In this case, mean reversion kicks in and the investor is wary about the negative consequences of a decrease in expected returns.

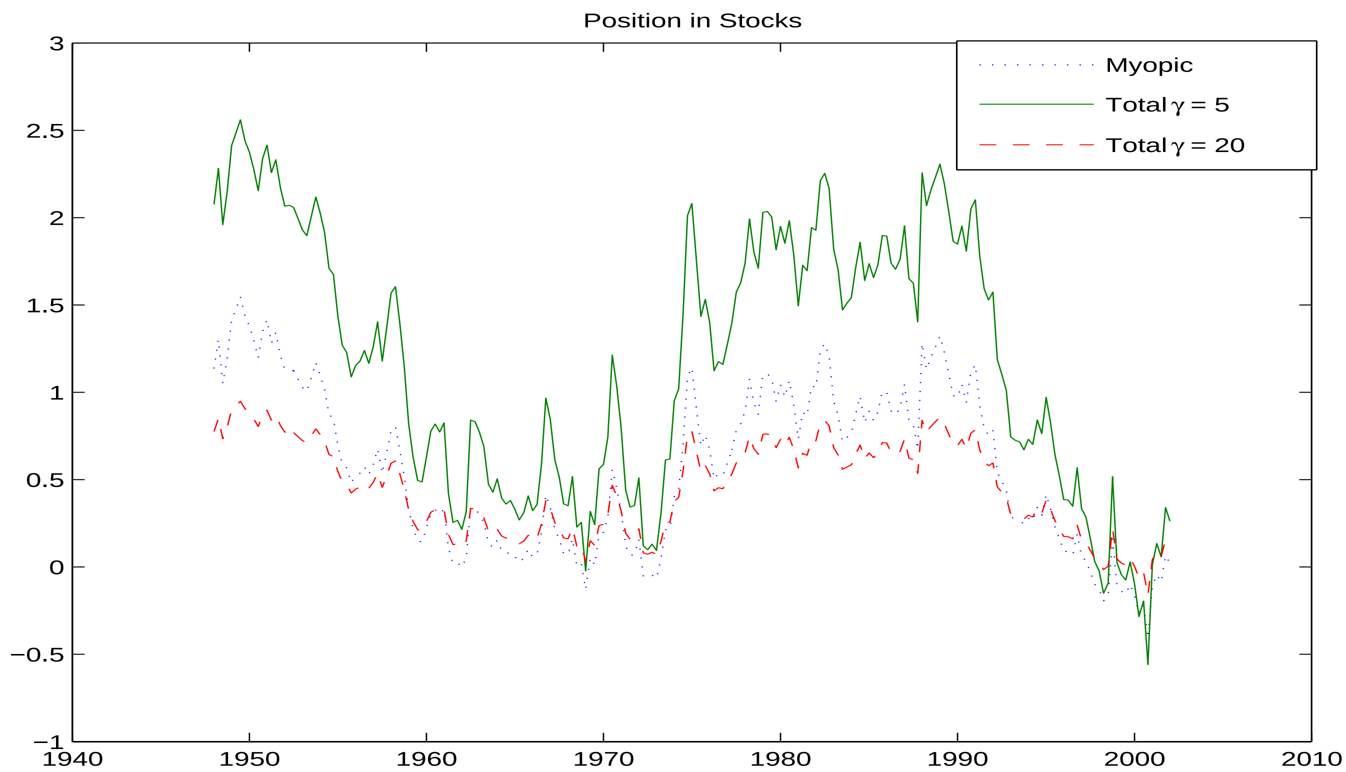
Total Demand over the Life Cycle



- Still, because of the hedging demand, an investor with 5 years to live would be still substantially exposed to stocks.

Strategic Asset Allocation over Time

- What is the variation over time of the optimal allocation to stock?
- Consider investor with $T = 15$ (constant) and $\gamma = 1, 5, 20$.



- The pattern for $\gamma = 20$ seems more reasonable than $\gamma = 1$ or 5.

Strategic Asset Allocation: Discussion

1. The predictability of stock returns is still source of heated debate.
   - Here we take the strong view that investors take empirical estimates as “true” parameters.
   - Much recent literature tried to relax this assumption, and use Bayesian methods in portfolio allocation
     * Kandel and Stambaugh (JF, 1997), Barberis (JF, 2000), Pastor (JF, 2000), Xia (JF, 2001).
     * These methodologies are very numerically intensive.

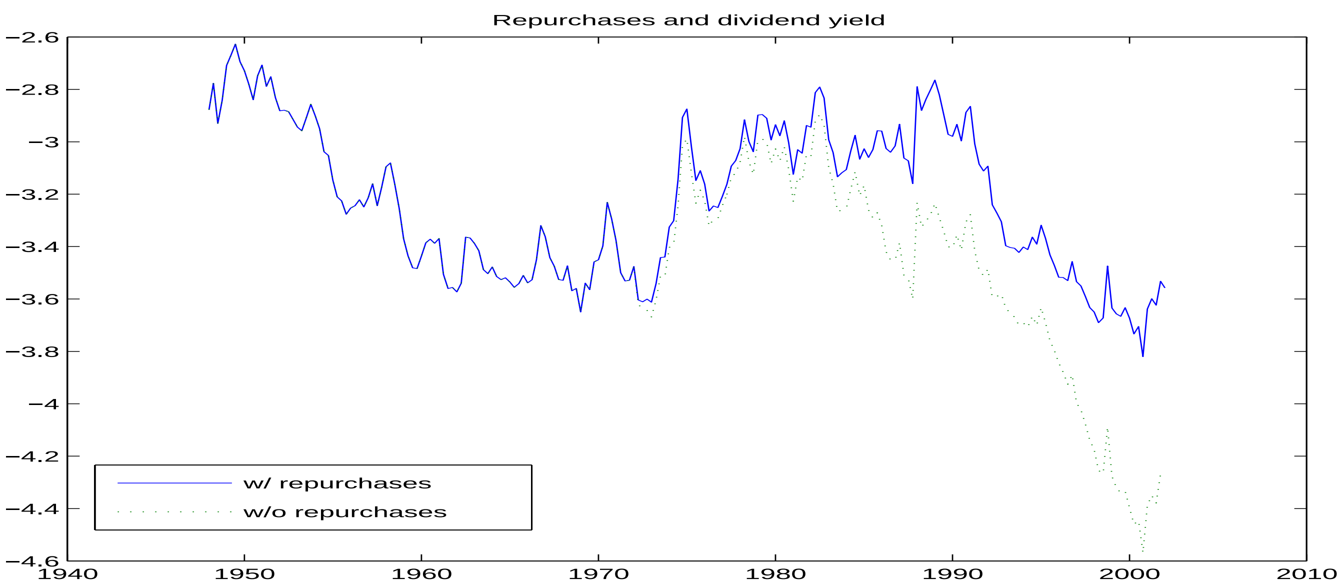
Strategic Asset Allocation: Discussion

2. As shown in Menzly, Santos and Veronesi (JPE, 2004), the dividend yield in which dividends are corrected for stock repurchases is a superior forecaster of future returns that the traditional dividend yield.

   – Without repurchases we have

Sample: 1947 - 2001. $dt = .25$

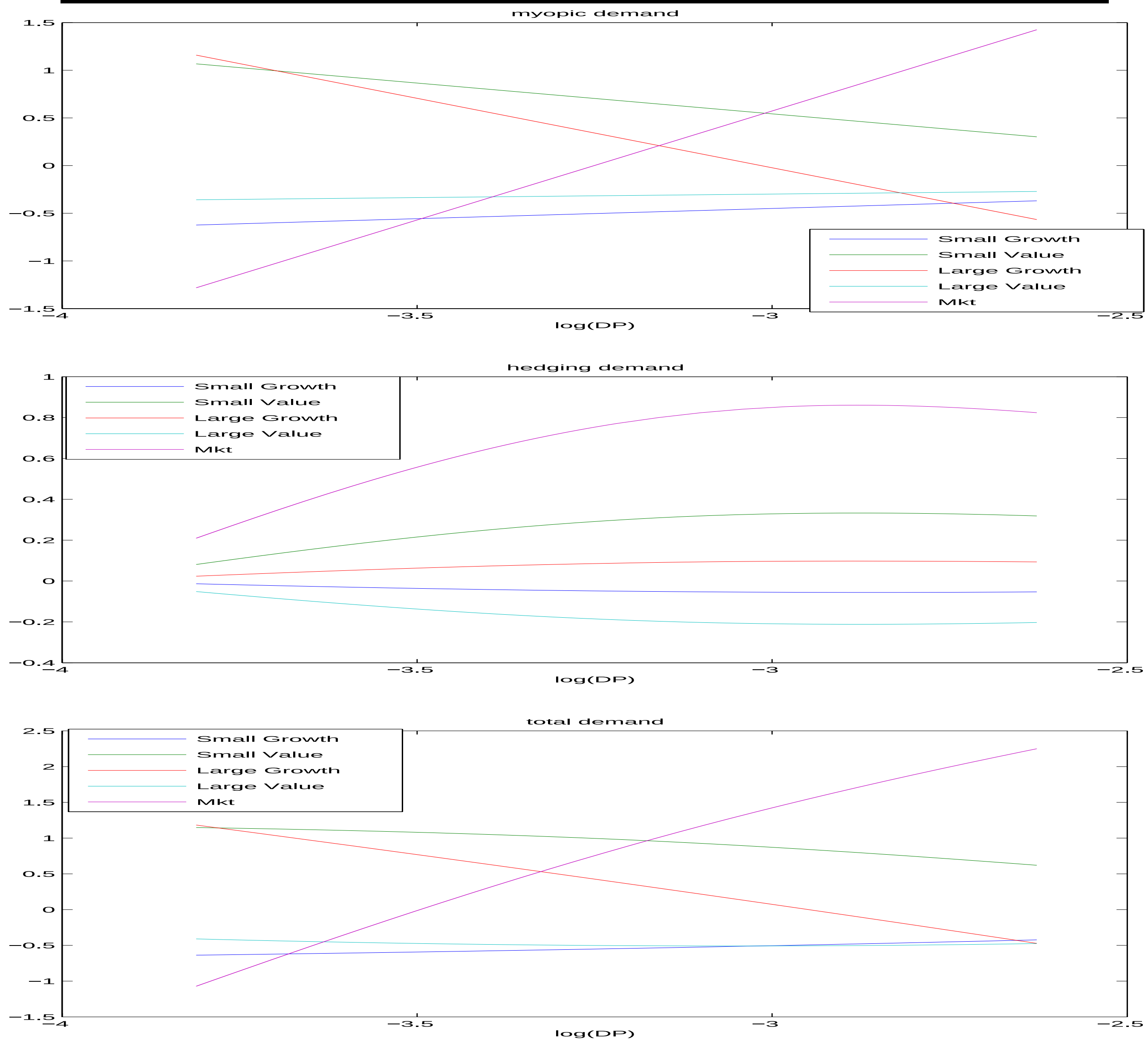
$\tilde{\beta}_0$	t-stat	$\tilde{\beta}_1$	t-stat	R^2
0.1233	2.5376	0.0310	2.0815	2.24%



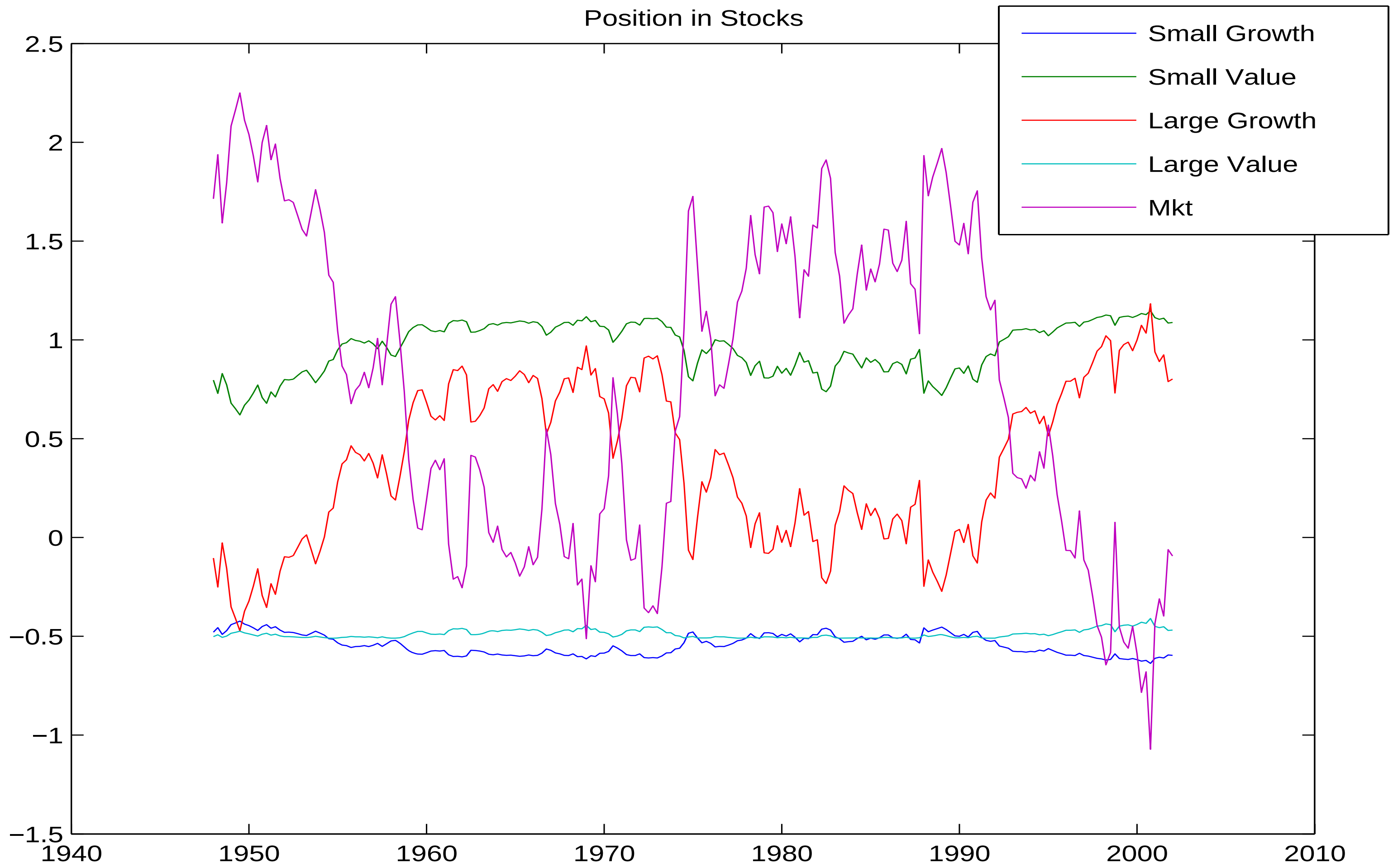
Strategic Asset Allocation: Discussion

3. The setting above can be easily extented to multiple assets and multiple predictors.
   - Analytical solutions are quite useful in this case.
   - Most models of strategic asset allocation do not go over the two or three assets.
   - As an illustration, next pictures show the strategic asset allocation for an investor who in addition to a market index, he has access to the returns from mutual funds specialized in 4 strategies:
     1. Value / Small Cap
     2. Value / Large Cap
     3. Growth / Small Cap
     4. Growth / Large Cap

Allocation to 6 Size - BM sorted portfolios and market



Allocation to 6 Size - BM sorted portfolios and market



Application 2: Learning about Average Returns

- Consider the same setting as in the original Merton problem

$$d\mathbf{R}_t = \boldsymbol{\mu} dt + \boldsymbol{\sigma} d\mathbf{B}_t$$

- Differently from Merton, assume that average returns $\boldsymbol{\mu}$ are not observable.
- Investors observe *realized* returns $d\mathbf{R}_t$ and infer the value of $\boldsymbol{\mu}$.
- Since the risk free rate r is observable, we can equivalently assume that agents infer the value of the average excess return $\boldsymbol{\lambda} = \boldsymbol{\mu} - r\mathbf{1}_n$.
- The following filtering result holds.

A Filtering Result

- Result: Let investors prior distribution at time 0 on $\boldsymbol{\lambda}$ be given by

$$\boldsymbol{\lambda}|_{t_0} \sim N\left(\widehat{\boldsymbol{\lambda}}_0, \widehat{\mathbf{q}}_0\right)$$

- Then, the posterior distribution at any time t is given by

$$\boldsymbol{\lambda}|_{t} \sim N\left(\widehat{\boldsymbol{\lambda}}_t, \widehat{\mathbf{q}}_t\right)$$

- where

$$\begin{aligned} d\widehat{\boldsymbol{\lambda}}_t &= \widehat{\boldsymbol{\Sigma}}_t d\widehat{\mathbf{B}}_t \\ \widehat{\boldsymbol{\Sigma}}_t &= \widehat{\mathbf{q}}_t \left(\boldsymbol{\sigma}'\right)^{-1} \\ \frac{d\widehat{\mathbf{q}}_t}{dt} &= -\widehat{\mathbf{q}}_t \left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1} \widehat{\mathbf{q}}_t \end{aligned}$$

- The innovation process is

$$d\widehat{\mathbf{B}}_t = \boldsymbol{\sigma}^{-1}\left[d\mathbf{R}_t - E_t\left(d\mathbf{R}_t\right)\right] \tag{1}$$

An Informational Equivalent Setting

- We can rewrite the system of returns then as follows

$$\begin{aligned} d\mathbf{R}_t &= \left(r + \widehat{\boldsymbol{\lambda}}\right) dt + \boldsymbol{\sigma} d\widehat{\mathbf{B}}_t \\ d\widehat{\boldsymbol{\lambda}}_t &= \widehat{\boldsymbol{\Sigma}}_t d\widehat{\mathbf{B}}_t \end{aligned}$$

- This is very similar to the previous case. Note the following:

  1. We are back to complete markets: Conditional on investors' information, the set of BMs that drive returns $d\mathbf{R}_t$ is the same that drive expected return $\widehat{\boldsymbol{\lambda}}_t$.
     - The reason is that the information filtration is generated by the return process $d\mathbf{R}_t$.
     - Thus, expected returns will depend on the observation of $d\mathbf{R}_t$ only: if we observe high returns we change our posterior to on expected future returns. That is, expected returns and realized returns become perfectly correlated.
     - $\Longrightarrow$ The asset allocation solution is exact!

An Informational Equivalent Setting

2. The only difference from the problem discussed earlier is the fact that the volatility of $\widehat{\boldsymbol{\lambda}}_t$ depends on t.
   - However, this volatility declines deterministically.
   - Thus, the methodology developed earlier applies here too, once we are careful to remember that $\widehat{\Sigma}_t$ is a function of time.

3. The volatility $\widehat{\Sigma}_t$ converges to zero as $t \to \infty$
   - This is because we assume $\boldsymbol{\lambda}$ is constant forever. Assuming some time variation in underlying average return will prevent the posterior variance from converging.
   - E.g. for the case $n = 1$,

$$q_t = \frac{1}{q_0^{-1} + \sigma^{-2} t}$$

An Informational Equivalent Setting

4. **Learning has a bite**: It has a prediction about the correlation between returns and expected returns.

$$Cov_t\left(d\mathbf{R}_t, d\boldsymbol{\lambda}_t\right) = \boldsymbol{\sigma}\widehat{\boldsymbol{\Sigma}}_t' = \boldsymbol{\sigma}\left(\boldsymbol{\sigma}\right)^{-1}\widehat{\mathbf{q}}_t = \widehat{\mathbf{q}}_t$$

– They are *p*ositively correlated: A negative innovation in returns decreases expected return.

– The hedging demand will go in the right direction here:

Bad news on returns are "twice bad news". You lost money, and now you expect to gain even less in the future.

– This is opposite of what we found in our earlier exercise, where we used the "predictability" intuition: negative returns increases the dividend price ratio, which predicts higher returns. That is, realized returns and expected returns were negatively correlated.

An Equivalent Portfolio Problem

- **Investor problem**:

$$J\left(W_0, \widehat{\lambda}_0, 0\right) = \max_{\{(C_t),(\boldsymbol{\theta}_t)\}} E_0\left[\int_0^T u\left(C_t, t\right) dt\right]$$

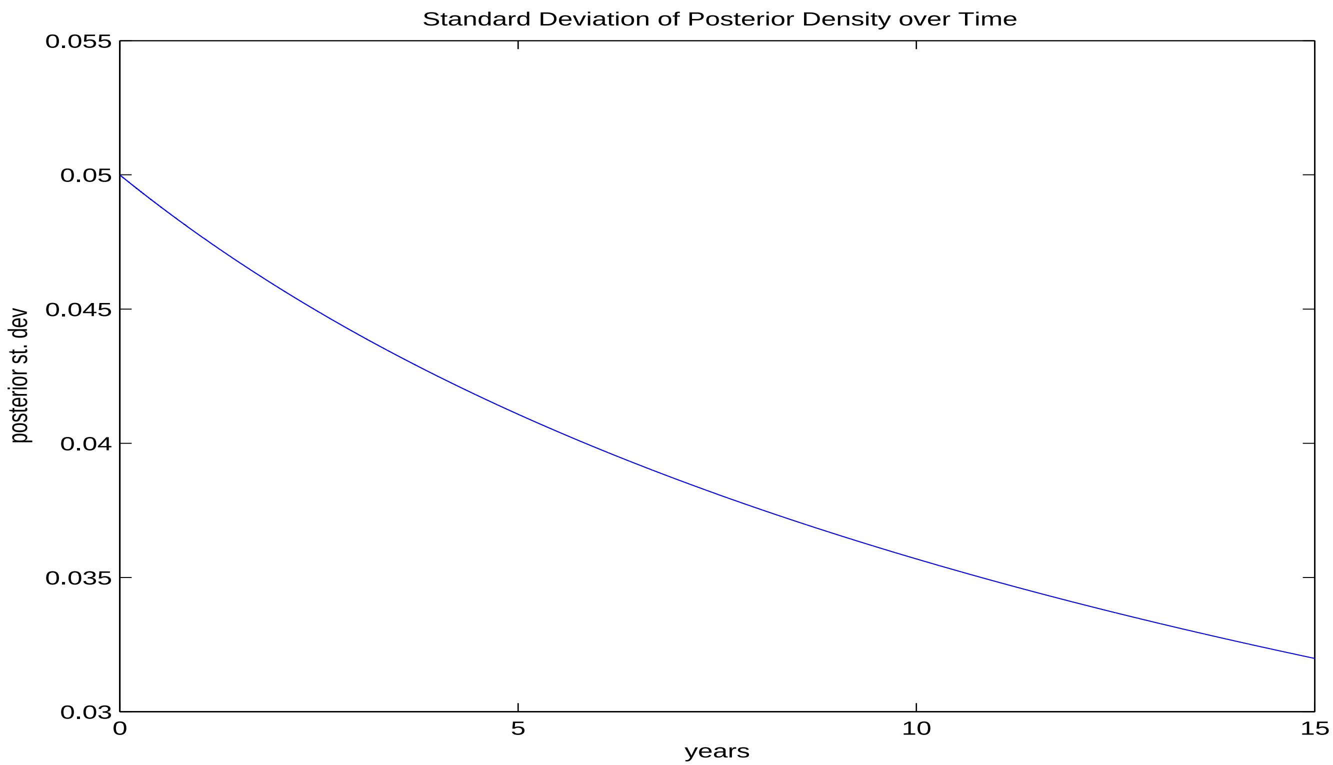
- subject to

$$dW_t = \left\{W_t\left(\boldsymbol{\theta}_t'\widehat{\boldsymbol{\lambda}}_t + r\right) - C_t\right\} dt + W_t\boldsymbol{\theta}_t'\boldsymbol{\sigma} d\mathbf{B}_t$$

- At this point, the solution is "almost" the same as before.
  - We need to set $\mathbf{A}_0 = \mathbf{A}_1 = 0$
  - Remember that $\widehat{\boldsymbol{\Sigma}}_t$ depends on time t.
    - ∗ The computation is in fact straightforward, as we can simply iterate forward the ODE that defines $\widehat{\mathbf{q}}_t$ (Riccati equation)

How Fast Would an Investor Learn?

- First, how fast does “uncertainty” declines?
  - From a prior uncertainty $\sqrt{q_0} = 5\%$, it declines rather slowly.

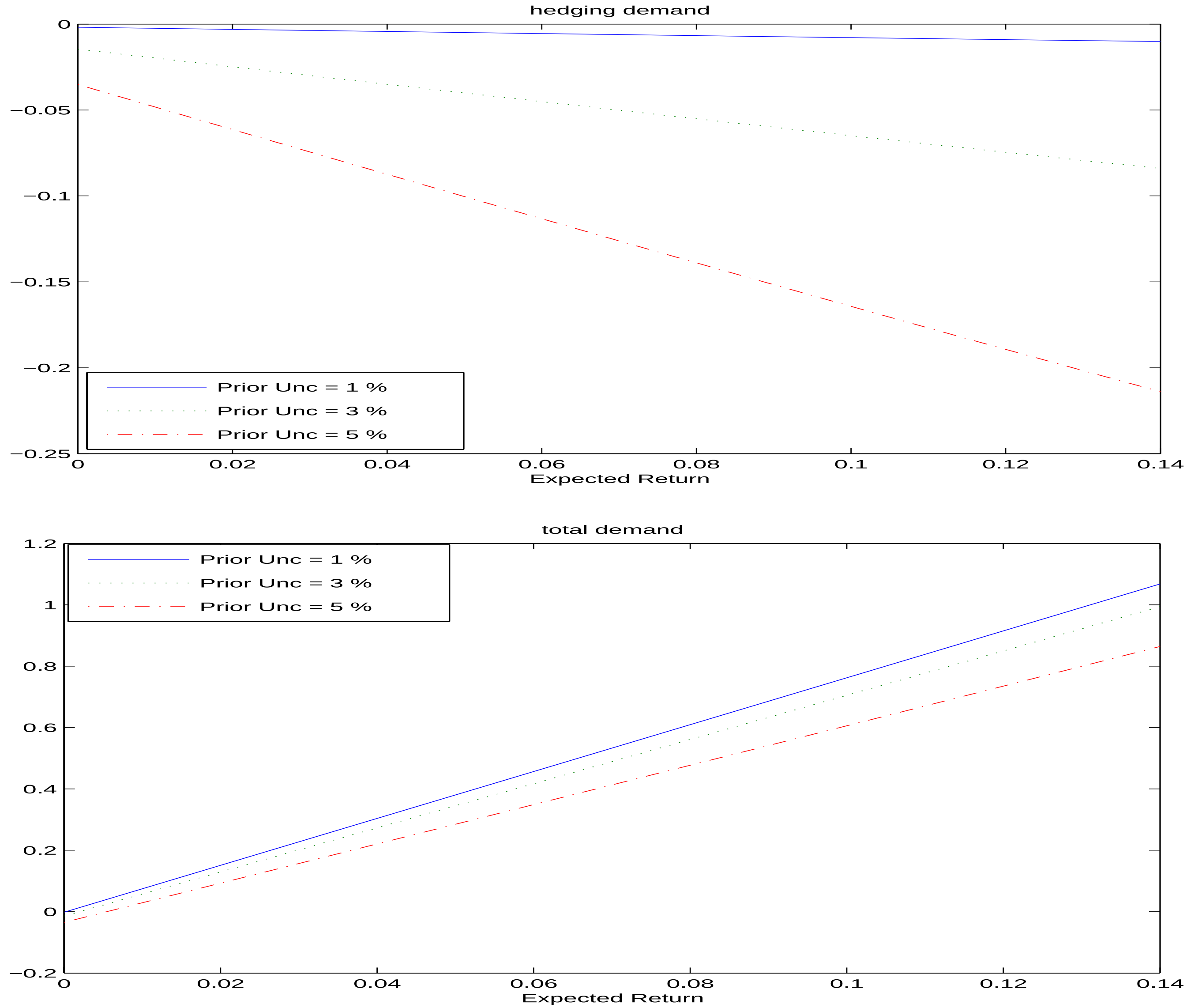


Strategic Asset Allocation with Learning: The Role of Prior Uncertainty

- The most important effect of learning is that hedging demand this time is negative.
- The intuition, recall, is that bad news are twice bad news here:
  - not only you get a negative return to stock, but now you expected even lower returns for the future.
  - Thus, investors' optimally reduce their holding of stocks.
  - This mechanism was first observed by Brennan (1998, European Finance Review), but then analyzed by many others.
- The following figures show the hedging demand and total demand for three different value of initial uncertainty

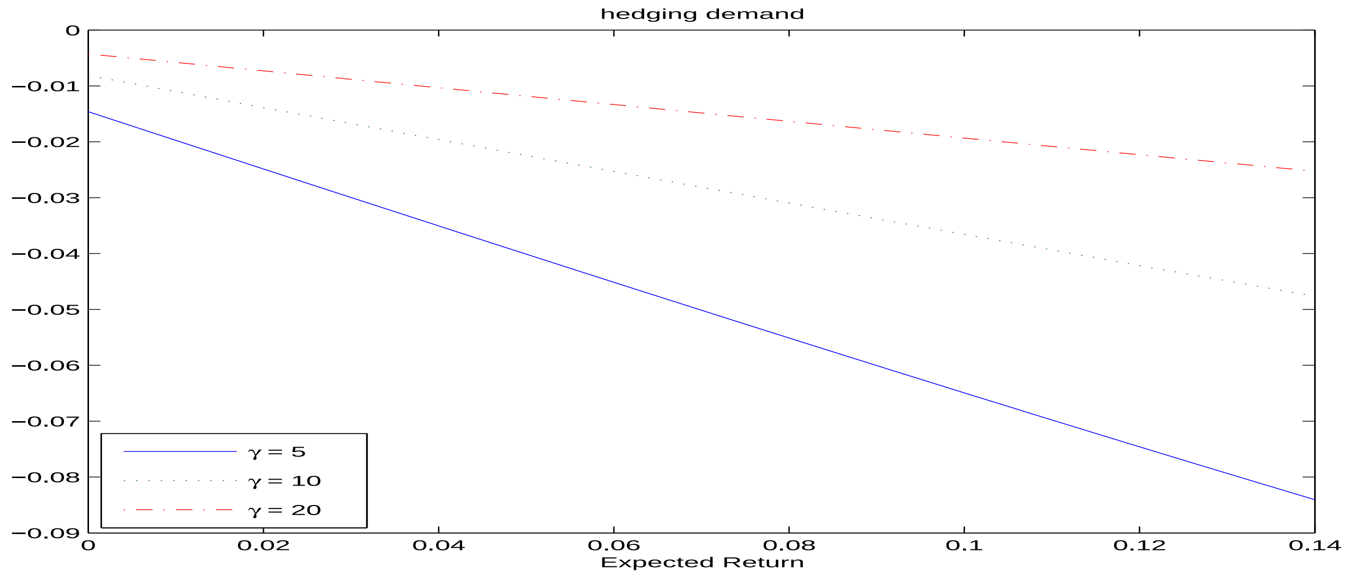
$$\sqrt{\hat{q}_0} = 1\%, 3\%, 5\%$$

Strategic Asset Allocation with Learning: The Role of Prior Uncertainty



Strategic Asset Allocation with Learning: The Role of Risk Aversion

- What effect does risk aversion have on hedging demands?



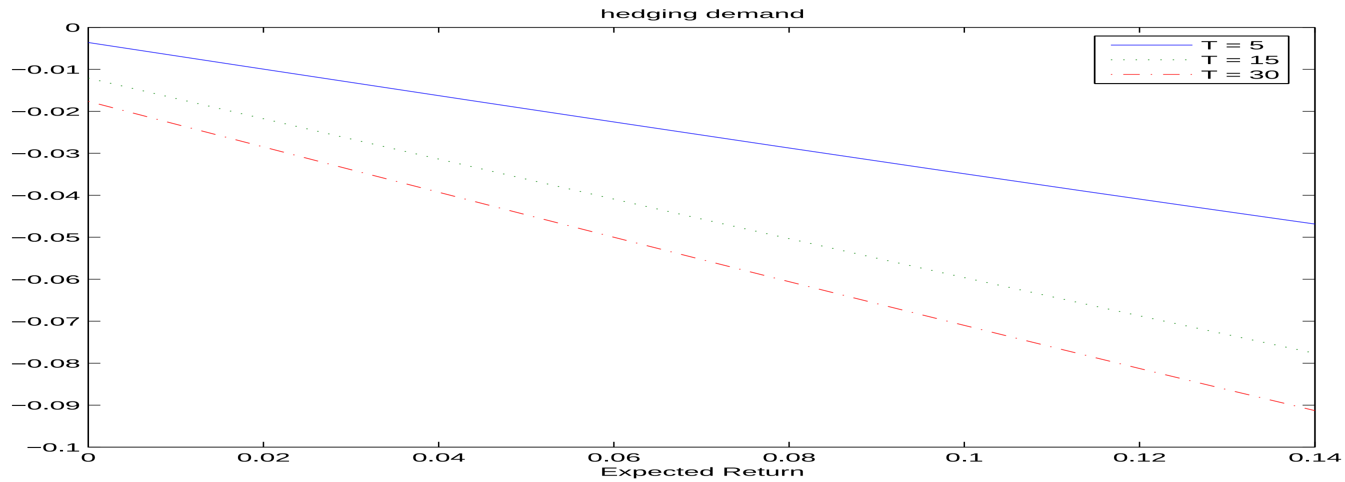
- Higher risk aversion decreases (in absolute value) the hedging demand.

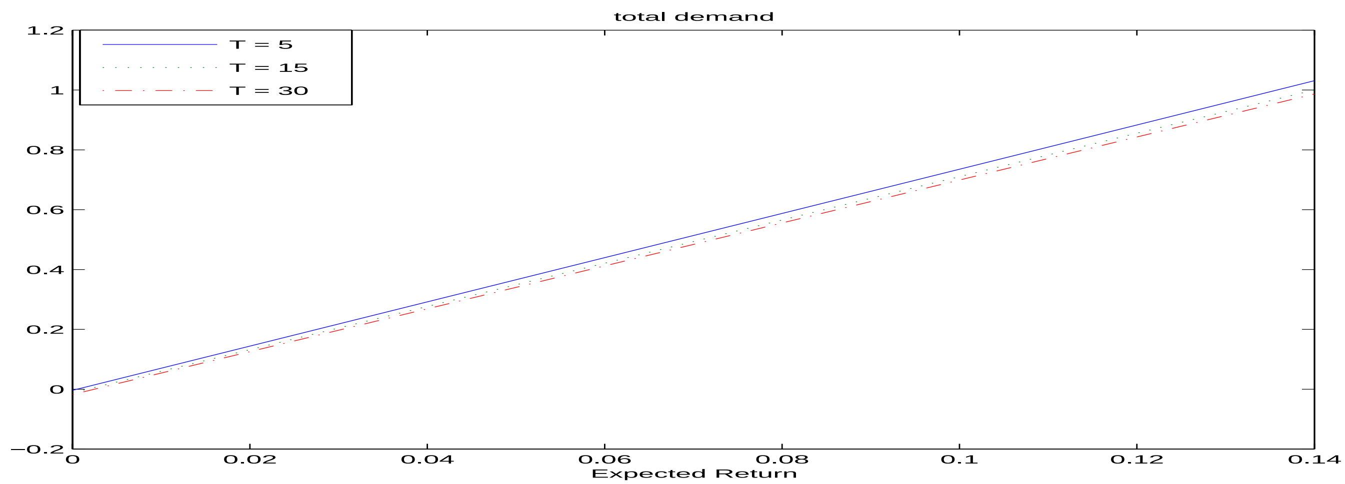
Strategic Asset Allocation with Learning: The Role of Risk Aversion

- Why does higher risk aversion decreases (in absolute value) the hedging demand?
  - This is due to the sensitivity of the consumption / wealth ratio C/W to changes in expected returns.
  - As we increase γ, the myopic demand for stocks decreases.
    - $\Longrightarrow$ The consumption to wealth ratio C/W becomes more and more insensitive to variation expected return.
    - $\Longrightarrow$ Eventually, changes in expected return have no impact on C/W, and thus no need of hedging demand.
    - $\Longrightarrow$ The relation between γ and hedging demand is non-linear, as hedging demand are close to zero both for γ close to 1 and for γ large.

Strategic Asset Allocation with Learning: The Life Cycle Implications

- How does learning affect the allocation of investors with different life expectancies?





Strategic Asset Allocation with Learning: The Life Cycle Implications

- Learning does not seem to have a large impact on the asset allocation as a function of time T.
- The little that is has goes in the opposite direction:
  - The reason, again, is the EIS.
  - The longer the horizon, the higher the impact of an increase in expected return on future consumption.
  - $\Longrightarrow$ larger decrease in θ_t due to consumption smoothing.

Strategic Asset Allocation with Learning over time

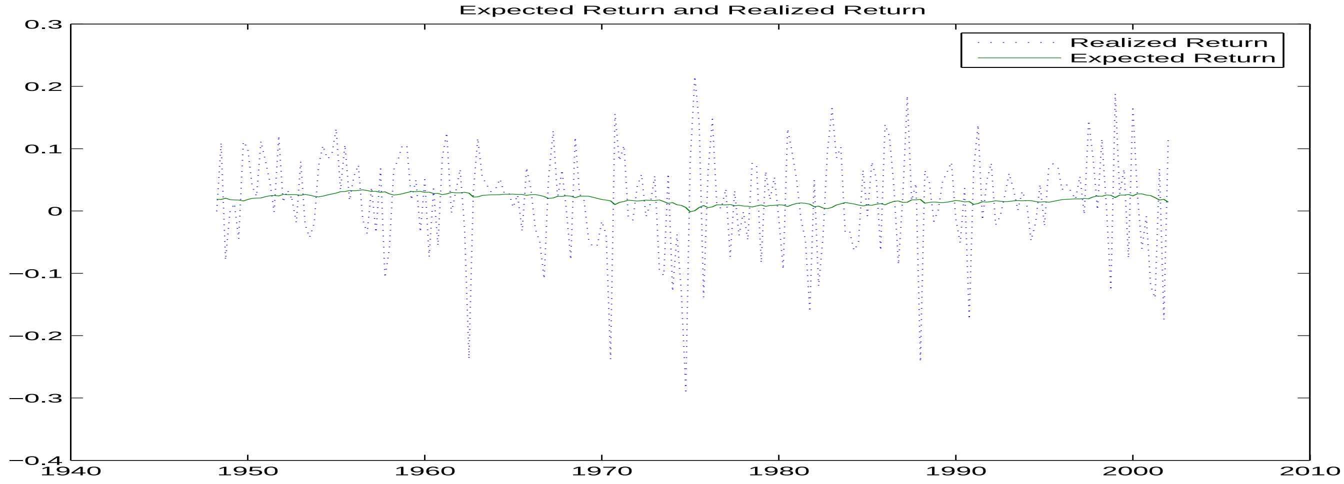
- Consider an investor in 1947 with prior uncertainty $\sqrt{q_0} = 5\%$.
  - How would his asset allocation change over time?

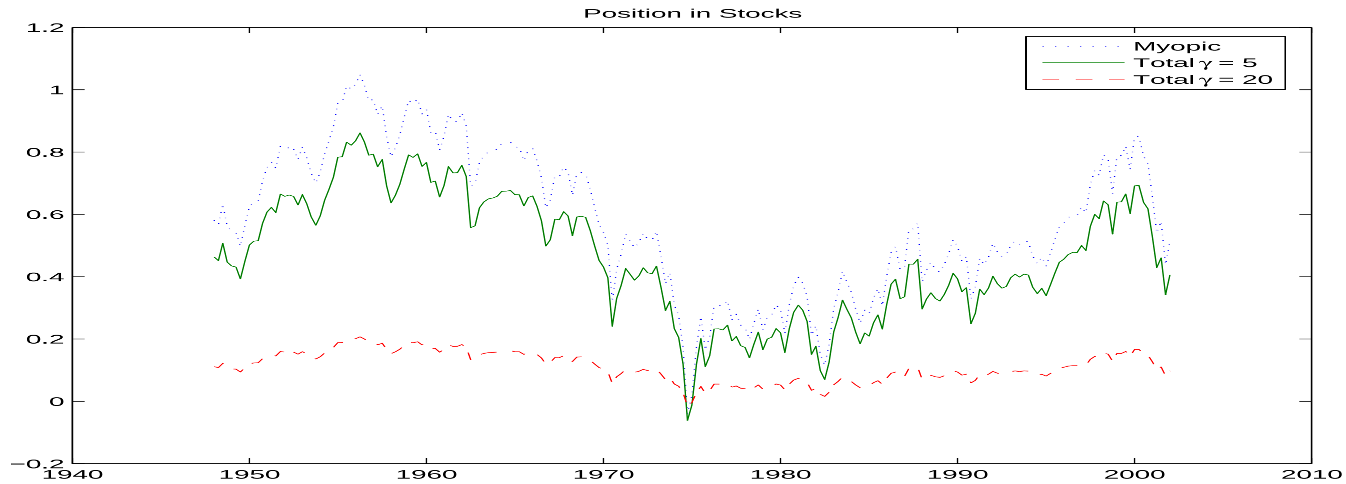
Strategic Asset Allocation with Learning over time

- Case 1: Assume a declining uncertainty over time

Strategic Asset Allocation with Learning over time

- Case 2: Assume a constant uncertainty (e.g. small probability of jumps)

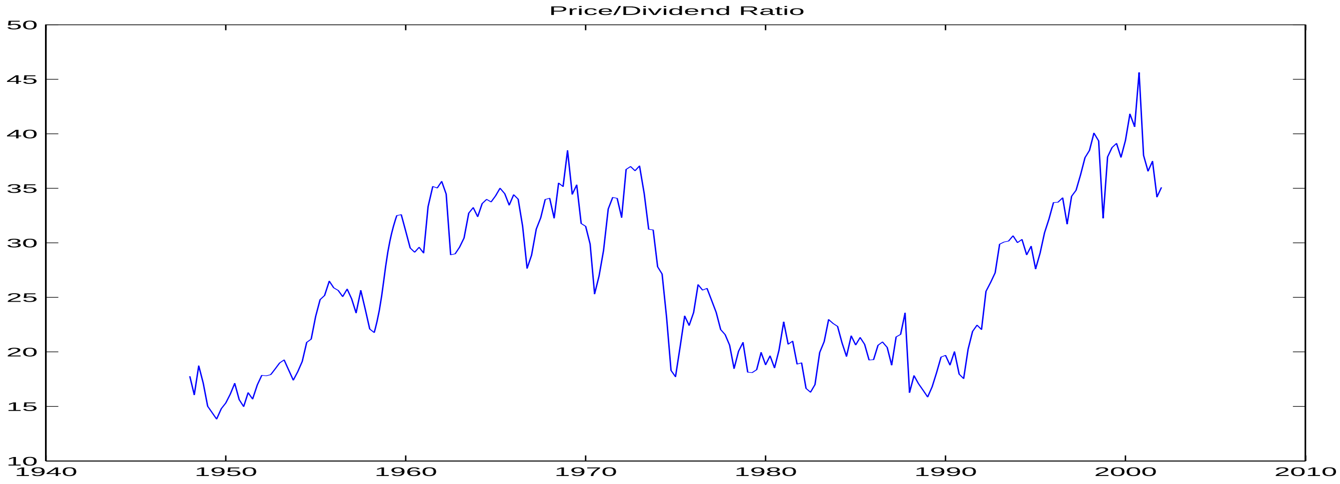


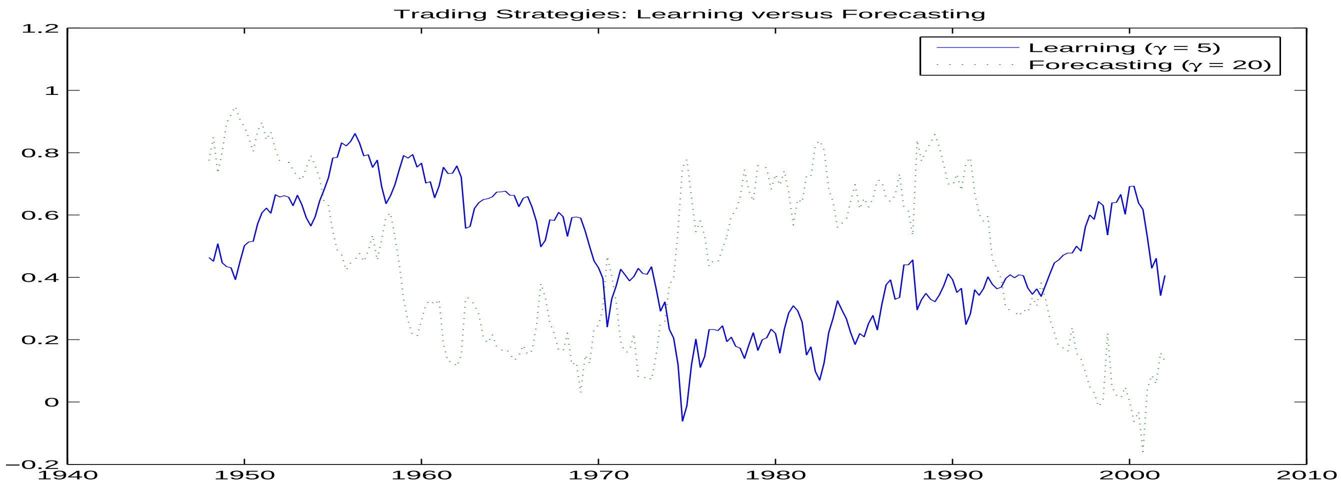


Strategic Asset Allocation and Expected Returns: Comparison

- Learning about average returns:
  - $\Longrightarrow$ Investor behave like "momentum" traders (or trend chasers)
    - * They buy when prices increase.
- Forecasting returns using the dividend yield:
  - $\Longrightarrow$ Investors behave like reversal traders
    - * They buy when prices drop

Strategic Asset Allocation and Expected Returns: Comparison





Strategic Asset Allocation with Model Misspecification

- What if investors are uncertain about the "model" and would like to take decisions that are "robust" to small misspecification?
  - We now discuss preferences for robustness and their implications for strategic portfolio allocation
  - The framework is the one of Anderson, Hansen, Sargent (ReStud 1999) as well as Maenhout (RFS, 2004)

- Consider (again!) the usual setting, with

$$d\mathbf{R}_t = (r + \boldsymbol{\lambda}_t)\, dt + \boldsymbol{\sigma} d\mathbf{B}_t$$

$$d\boldsymbol{\lambda}_t = (\mathbf{A}_0 + \mathbf{A}_1 \boldsymbol{\lambda}_t)\, dt + \boldsymbol{\Sigma} d\mathbf{B}_t$$

  - Let P denote the probability measure that is defined by these processes.
  - We call this the "reference model".

Modeling "Model Misspecification"

- The investor is worried about "small" model misspecification.
- Two questions:
  1. How can we model a model misspecification?
  2. How can we model investor "aversion" to such misspecification?

- We can model "model misspecification" by introducing a set of "plausible" probability measures Q that are "close" to the original one P.
- In continuous time, we can "perturb" the reference model and obtain new probability measures Q by replacing $d\mathbf{B}_t$ by

$$d\mathbf{B}_t = d\widehat{\mathbf{B}}_t + \mathbf{h}_t dt$$

- where $\mathbf{h}_t$ is another stochastic process.

Modeling "Model Misspecification"

- The class of misspecified models is then those defined by the

$$d\mathbf{R}_t = (r + \boldsymbol{\lambda}_t)\, dt + \boldsymbol{\sigma} \left(d\widehat{\mathbf{B}}_t + \mathbf{h}_t dt\right)$$

$$d\boldsymbol{\lambda}_t = (\mathbf{A}_0 + \mathbf{A}_1 \boldsymbol{\lambda}_t)\, dt + \boldsymbol{\Sigma} \left(d\widehat{\mathbf{B}}_t + \mathbf{h}_t dt\right)$$

- for "plausible" $\mathbf{h}_t$ processes.
- How can we introduce "preferences" for robustness?

Modeling "Model Misspecification"

- The *multiplier robust control problem* can be formulated as

$$\sup_{C,\boldsymbol{\theta}} \inf_{\mathbf{h}} \left\{ \widehat{E} \left[\int_0^T e^{-\rho t} \left(u\left(C_t\right) + \frac{\eta}{2} \mathbf{h}_t \mathbf{h}_t' \right) dt \right] \right\}$$

  – subject to the "perturbed" budged equations

$$dW_t = \left(W_t \left(\boldsymbol{\theta}_t' \boldsymbol{\lambda}_t + r \right) - C_t \right) dt + W_t \boldsymbol{\theta}_t' \boldsymbol{\sigma} \left(d\widehat{\mathbf{B}}_t + \mathbf{h}_t dt \right)$$

- Here η is a penalty imposed on the discrepancy between Q and P.
- For given η, the "robust" investor
  1. considers the probabilities Q (each defined by a process $\mathbf{h}_t$) that lead to low utility ($\inf_{\mathbf{h}}$ part)
  2. maximizes utility taking into account these worst case scenarios ($\max_{C,\boldsymbol{\theta}}$ part)

Modeling "Model Misspecification"

- A high η implies a choice of $\mathbf{h}_t$ that is close to 0, i.e. a probability Q that is close to P, because we are taking the "inf" with respect to $\mathbf{h}_t$.
  - If $\eta = 0$, we consider all the possible Q's.
  - If $\eta = \infty$, we consider only P.

Strategic Asset Allocation with Model Misspecification

- How can we solve this "max min" problem?
- It is convenient to stack all the state variables. Define $\mathbf{Y}_t = \left(W_t, \boldsymbol{\lambda}_t'\right)'$, so that we have

$$d\mathbf{Y}_t = \boldsymbol{\mu}_Y\left(\mathbf{Y}_t, \boldsymbol{\theta}_t, C_t\right)dt + \boldsymbol{\sigma}_Y\left(\mathbf{Y}_t, \boldsymbol{\theta}_t, C_t\right)\left(d\widehat{\mathbf{B}}_t + \mathbf{h}_t dt\right)$$

- The following Bellman Isaac condition is the necessary condition for the solution to the max min problem
- There exists a value function $J(Y)$ such that

$$\delta J = \max_{C,\boldsymbol{\theta}} \min_{\mathbf{h}} \left\{ u(C) + \frac{\eta}{2}\mathbf{h}\mathbf{h}' + \left(\mu_Y + \sigma_Y \mathbf{h}'\right)' \mathbf{J}_Y + \frac{1}{2} tr\left(\boldsymbol{\sigma}_Y' \mathbf{J}_{YY} \boldsymbol{\sigma}_Y'\right) \right\}$$

Towards a Solution to the Asset Allocation

- Solving for the minimum $\mathbf{h}$, one obtains

$$\mathbf{h}' = -\frac{1}{\eta}\boldsymbol{\sigma}_Y'\mathbf{J}_Y$$

- Notice that then

$$\begin{aligned}\frac{\eta}{2}\mathbf{h}\mathbf{h}' &= \frac{1}{2\eta}\mathbf{J}_Y'\boldsymbol{\sigma}_Y\boldsymbol{\sigma}_Y'\mathbf{J}_Y \\ \boldsymbol{\sigma}_Y\mathbf{h}' &= -\frac{1}{\eta}\boldsymbol{\sigma}_Y\boldsymbol{\sigma}_Y'\mathbf{J}_Y\end{aligned}$$

- Substitute into Bellman Isaac equation to find

$$\delta J = \max_{C,\boldsymbol{\theta}}\left\{u(C) - \frac{1}{2\eta}\mathbf{J}_Y'\sigma_Y\sigma_Y'\mathbf{J}_Y + \mu_Y'\mathbf{J}_Y + \frac{1}{2}tr\left(\sigma_Y'\mathbf{J}_{YY}\boldsymbol{\sigma}_Y\right)\right\}$$

- This is similar to earlier problem.

Optimal Consumption and Asset Allocation under Model Misspecification

- The FOC with respect to consumption lead to the usual condition

$$u_C = J_W$$

  - But J_W is different from before. It will depend on robustness preferences

- Instead, the FOC for optimal portfolio weights imply

$$\begin{aligned}
\boldsymbol{\theta}_t &= \frac{-J_W}{W_t\left(J_{WW} - \frac{1}{\eta}J_W^2\right)}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\left(\boldsymbol{\lambda}_t\right) \\
&+ \frac{-1}{W_t\left(J_{WW} - \frac{1}{\eta}J_W^2\right)}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\sigma}\boldsymbol{\Sigma}'\mathbf{J}_{W\boldsymbol{\lambda}} \\
&+ \frac{\frac{1}{\eta}J_W}{W_t\left(J_{WW} - \frac{1}{\eta}J_W^2\right)}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\sigma}\boldsymbol{\Sigma}'\mathbf{J}_{\boldsymbol{\lambda}}
\end{aligned}$$

Strategic Asset Allocation under Model Misspecification

- The portfolio rule has then three components:

1. Standard myopic demand.
   - Notice that the denominator is adjusted for robustness, implying a lower investment in the stocks (because $J_W^2 \frac{1}{\eta} > 0$).

2. The standard Merton's hedging demand.

3. An additional hedging demand arising from robustness preferences.
   - If $\eta \to \infty$, i.e. we consider the class of probability Q that are closer and closer to the reference P, we have back the usual results.
   - Note in particular that the last term drops out.

An Exact Solution for the Original Merton Problem

- Consider the original setting without time varying expected returns.
  - i.e. $\mathbf{A}_0 = 0$, $\mathbf{A_1} = 0$ and $\boldsymbol{\Sigma} = 0$

- In this case, the FOC with respect to $\mathbf{h}_t$ yield

$$\mathbf{h}_t = -\frac{1}{\eta}\boldsymbol{\sigma}'_W J_W$$

- and the Bellman Isaac equation is then given by

$$\delta J = \max_{C,\boldsymbol{\theta}}\left\{u(C) - \frac{1}{2\eta}J_W^2\boldsymbol{\sigma}_W\boldsymbol{\sigma}'_W + \mu_W J_W + \frac{1}{2}J_{WW}\boldsymbol{\sigma}_W\boldsymbol{\sigma}'_W\right\}$$

- Using $u_c = J_W$ we obtain

$$\boldsymbol{\theta}_t = \frac{-J_W}{W_t\left(J_{WW} - \frac{1}{\eta}J_W^2\right)}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\left(\boldsymbol{\mu} - r\mathbf{1}_d\right)$$

An Exact Solution for the Original Merton Problem

- One complication with the previous problem is that, generically, it is not "scale invariant"
  - It is hard to solve as the solution depends on wealth.

- Maenheut (2004) proposes to scale the penalty parameter η by the value function J itself, in a way to make the model again scale independent.

$$\eta = \eta(J) = \eta^* (1-\gamma) J(W,t)$$

- The value function is then given by

$$J(W,t) = \left(\frac{1 - e^{-a(T-t)}}{a} \right)^{\gamma} \frac{W^{1-\gamma}}{1-\gamma}$$

- where

$$a = \frac{1}{\gamma} \left[\rho - (1-\gamma) r - \frac{1-\gamma}{2(\gamma+\eta)} (\boldsymbol{\mu} - r\mathbf{1}_n)' (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1} (\boldsymbol{\mu} - r\mathbf{1}_n) \right]$$

An Exact Solution for the Original Merton Problem

- The optimal consumption and asset allocation are

$$C_t = \frac{a}{1 - e^{-a(T-t)}} W_t$$

$$\boldsymbol{\theta}_t = \frac{1}{\gamma + 1/\eta^*} \left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1} \left(\boldsymbol{\mu} - r\mathbf{1}_d\right)$$

  - Preferences for robustness clearly go in the right direction to "solve" the asset allocation puzzle
  - A lower η^* translates into a higher "aversion" to model misspecification.
  - In this case, the allocation to stocks decreases.
  - Yet, the allocation is still independent of life expectancy $T - t$.
    - We need to introduce predictability for that.

How much pessimism is plausible?

- Clearly, by decreasing η^* we can match any empirically observed level of asset holdings.
- However, the question is then what is a "reasonable" level of η^*.
- Consider the case $n = 1$ (one stock) for simplicity.
  - For each level of η^*, there is a given worst case scenario, defined by the FOC

$$h_t = -\frac{1}{\eta}\sigma_W J_W = -\frac{1}{(1+\gamma\eta^*)\,\sigma}(\mu - r)$$

  - where I substitute for $\sigma_W = W\theta_t\sigma$, J_W and $\eta = \eta^*(1-\gamma)J$.
- A robust investor thinks that stock returns are given by

$$dR_t = (\mu + \sigma h_t)dt + \sigma d\widehat{B}_t$$

How much pessimism is plausible?

- Thus, the equity premium for a robust investor is

$$E_t^h[dR - r] = (\mu + \sigma h_t) - r = (\mu - r)\left(1 - \frac{1}{1 + \gamma\eta^*}\right)$$

- We can use the "implied" perceived equity premium of the robust investor as a reasonable metric to asset whether η^* is too small.

Optimal Portfolio Allocation under Robustness

	γ									
	2		4		6		8		10	
η	θ	$E^h[dR]$	θ	$E^h[dR]$	θ	$E^h[dR]$	θ	$E^h[dR]$	θ	$E^h[dR]$
0.1	22.79	1.17	19.53	2.00	17.09	2.63	15.19	3.11	13.67	3.50
0.2	39.06	2.00	30.38	3.11	24.86	3.82	21.03	4.31	18.23	4.67
0.5	68.36	3.50	45.57	4.67	34.18	5.25	27.34	5.60	22.79	5.83
1	91.15	4.67	54.69	5.60	39.06	6.00	30.38	6.22	24.86	6.36
2	109.38	5.60	60.76	6.22	42.07	6.46	32.17	6.59	26.04	6.67
10	130.21	6.67	66.69	6.83	44.83	6.89	33.76	6.91	27.07	6.93
100	136.04	6.97	68.19	6.98	45.50	6.99	34.14	6.99	27.32	6.99

Recent Applications of Robust Control

- The approach of robust control theory has found numerous applications in finance in recent times.

1. Liu, Pan and Wang (JF, 2005): uncertainty on rare events to explain options premia, along with the standard result on return equity premium.
2. Routledge and Zin (2004): rare events and market liquidity. $\Longrightarrow$ uncertainty aversion may lead agents not to trade after big market events.
3. Uppal and Wang (JF, 2003): extend the above model to the case of different aversions to uncertainty across assets.
   - For some assets there is less "ambiguity" about the probabilities.
   - Under-diversification: even a limited amount of aversion to uncertainty on some stocks $\Longrightarrow$ over-invest in those with less uncertainty aversion.
4. Boyle, Uppal, Wang (2005) use a similar setting to "explain" the over-investment in "own stock" puzzle.

Conclusion

- The last decade has seen a boom in research about optimal asset allocation.
- The groundwork set by Samuelson and Merton has found application only recently, as researchers were able to solve long-standing problems
  - The concept of hedging demands date back 30+ years
  - But only recently these hedging demands have been characterized in a quantitative fashion.
- Yet, we are still far from explaining all of the puzzles in a nice, convincing theory.
  - Predictability has the right implication for life cycle, but wrong for asset allocation magnitudes
  - Learning has the right implication for the magnitudes, but wrong for life cycle
  - Preferences for robustness imply unreasonable levels of pessimism.