What Ties Return Volatilities to Price Valuations and Fundamentals?
On-Line Appendix

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This online appendix contains the complete proof to Proposition 1 (a), as well as additional material that is excluded from the article.

1. Proof of part (a) of Proposition 1

The proof follows the steps of Proposition 1 in Veronesi (2000). The pricing equation is

$$P_t = D_t \mathbb{E}\left[\int_t^\infty \frac{M_\tau D_\tau}{M_t D_t} d\tau\right] = D_t \mathbb{E}\left[\int_t^\infty \frac{M_\tau E_\tau}{M_t E_t} d\tau\right] = D_t \mathbb{E}\left[\int_t^\infty e^{(m_\tau - m_t)+(e_\tau - e_t)} d\tau\right]$$

where we assume that the usual transversality condition holds and thus all integrals are well defined.[1] From Ito's Lemma

$$\begin{aligned} dm_t &= \left(-k^i - \frac{\sigma_M \sigma_M'}{2}\right) dt - \sigma_M dW_t \\ de_t &= \left(\theta^i - \frac{\sigma_E \sigma_E'}{2}\right) dt + \sigma_E dW_t \end{aligned}$$

Denote

$$V\left(t;\nu^i\right) = \mathbb{E}\left[\int_t^\infty e^{(m_\tau - m_t)+(e_\tau - e_t)} d\tau | \nu_t = \nu^i\right]$$

Because the stochastic variable ν_t is right-continuous, we can select a small interval Δ such that $\nu_\tau = \nu^i$ for $\tau \in [t, t+\Delta)$, which yields

$$\begin{aligned} V\left(t;\nu^i\right) &= \mathbb{E}\left[\int_t^{t+\Delta} e^{(m_\tau - m_t)+(e_\tau - e_t)} d\tau | \nu_t = \nu^i\right] + \mathbb{E}\left[\int_{t+\Delta}^\infty e^{(m_\tau - m_t)+(e_\tau - e_t)} d\tau | \nu_t = \nu^i\right] \\ &= \mathbb{E}\left[\int_t^{t+\Delta} e^{\left(-k^i+\theta^i-\frac{\sigma_M\sigma_M'}{2}-\frac{\sigma_E\sigma_E'}{2}\right)(\tau-t)+(-\sigma_M+\sigma_E)(W_\tau - W_t)} d\tau | \nu_t = \nu^i\right] \\ &\quad + \mathbb{E}\left[\int_{t+\Delta}^\infty e^{(m_\tau - m_t)+(e_\tau - e_t)} d\tau | \nu_t = \nu^i\right] \end{aligned}$$

The first expectation is given by

$$\begin{aligned} & \mathbb{E}\left[\int_t^{t+\Delta} e^{\left(-k^i+\theta^i-\frac{\sigma_M\sigma_M'}{2}-\frac{\sigma_E\sigma_E'}{2}\right)(\tau-t)+(-\sigma_M+\sigma_E)(W_\tau - W_t)} d\tau | \nu_t = \nu^i\right] \\ =& \int_t^{t+\Delta} e^{\left(-k^i+\theta^i-\frac{\sigma_M\sigma_M'}{2}-\frac{\sigma_E\sigma_E'}{2}\right)(\tau-t)+\frac{1}{2}(-\sigma_M+\sigma_E)(-\sigma_M+\sigma_E)'(\tau-t)} d\tau \\ =& \int_t^{t+\Delta} e^{\left(-k^i+\theta^i-\sigma_M\sigma_E'\right)(\tau-t)} d\tau \\ =& \frac{e^{\left(-k^i+\theta^i-\sigma_M\sigma_E'\right)\Delta} - 1}{\left(-k^i+\theta^i-\sigma_M\sigma_E'\right)} \end{aligned}$$

[1] A sufficient condition is that the transversality condition holds for each regime, i.e. under the assumption of no regime shifts. In this case, the integral is finite if $-k^i + \theta^i - \sigma_M\sigma_E' < 0$.

The second term can also be computed as

$$\begin{aligned}
\mathbb{E}\left[\int_{t+\Delta}^{\infty} e^{(m_\tau - m_t)+(e_\tau - e_t)} d\tau | \nu_t = \nu^i\right] &= E\left[e^{(m_{t+\Delta}-m_t)+(e_{t+\Delta}-e_t)} \int_{t+\Delta}^{\infty} e^{(m_\tau - m_{t+\Delta})+(e_\tau - e_{t+\Delta})} d\tau | \nu_t = \nu^i\right] \\
&= \mathbb{E}\left[e^{(m_{t+\Delta}-m_t)+(e_{t+\Delta}-e_t)} | \nu_t = \nu^i\right] \times \\
&\quad \times \mathbb{E}\left[\int_{t+\Delta}^{\infty} e^{(m_\tau - m_{t+\Delta})+(e_\tau - e_{t+\Delta})} d\tau | \nu_t = \nu^i\right]
\end{aligned}$$

The first term in the last expression is

$$\mathbb{E}\left[e^{(m_{t+\Delta}-m_t)+(e_{t+\Delta}-e_t)} | \nu_t = \nu^i\right] = e^{\left(-k^i + \theta^i - \sigma_M \sigma_E'\right)\Delta}$$

and the second term can written as

$$\begin{aligned}
\mathbb{E}\left[\int_{t+\Delta}^{\infty} e^{(m_\tau - m_{t+\Delta})+(e_\tau - e_{t+\Delta})} d\tau | \nu_t = \nu^i\right] &= \sum_{j \neq i} E\left[\int_{t+\Delta}^{\infty} e^{(m_\tau - m_{t+\Delta})+(e_\tau - e_{t+\Delta})} d\tau | \nu_{t+\Delta} = \nu^j\right] \lambda_{ij}\Delta \\
&\quad + \mathbb{E}\left[\int_{t+\Delta}^{\infty} e^{(m_\tau - m_{t+\Delta})+(e_\tau - e_{t+\Delta})} d\tau | \nu_{t+\Delta} = \nu^i\right] \left(1 - \sum_{j \neq i} \lambda_{ij}\Delta\right) \\
&= \sum_{j \neq i} V\left(t+\Delta; v^j\right) \lambda_{ij}\Delta + V\left(t+\Delta; v^i\right)\left(1 - \sum_{j \neq i} \lambda_{ij}\Delta\right)
\end{aligned}$$

Using a Taylor expansion on time t, we see that for each i

$$V\left(t+\Delta; v^i\right) = V\left(t; v^i\right) + V'\left(t; v^i\right)\Delta + o\left(\Delta\right)$$

Putting all the terms together, and eliminating terms $o\left(\Delta^n\right)$ for $n > 1$, we obtain

$$\begin{aligned}
V\left(t; v^i\right) &= \frac{e^{\left(-k^i + \theta^i - \sigma_M \sigma_E'\right)\Delta} - 1}{\left(-k^i + \theta^i - \sigma_M \sigma_E'\right)} \\
&\quad + e^{\left(-k^i + \theta^i - \sigma_M \sigma_E'\right)\Delta}\left[V\left(t; v^i\right) + V'\left(t; v^i\right)\Delta - \sum_{j \neq i} \lambda_{ij} V\left(t; v^i\right)\Delta + \sum_{j \neq i} \lambda_{ij} V\left(t; v^j\right)\Delta\right]
\end{aligned}$$

Rearranging, dividing by Δ, and recalling $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$, we obtain

$$V\left(t; v^i\right)\left(\frac{1 - e^{\left(-k^i + \theta^i - \sigma_M \sigma_E'\right)\Delta}}{\Delta}\right) = \frac{e^{\left(-k^i + \theta^i - \sigma_M \sigma_E'\right)\Delta} - 1}{\left(-k^i + \theta^i - \sigma_M \sigma_E'\right)\Delta} + e^{\left(-k^i + \theta^i - \sigma_M \sigma_E'\right)\Delta}\left[V'\left(t; v^i\right) + \sum_{j=1}^{n} \lambda_{ij} V\left(t; v^j\right)\right]$$

Taking the limit as $\Delta \to 0$, and rearranging

$$V'\left(t; v^i\right) = -1 - \sum_{j=1}^{n} \lambda_{ij} V\left(t; v^j\right) + \left(k^i - \theta^i + \sigma_M \sigma_E'\right) V\left(t; v^i\right)$$

Define

$$\mathbf{A} = -\mathbf{\Lambda} + diag\left(k^i - \theta^i + \sigma_M \sigma_E'\right)$$

and write in vector form

$$\mathbf{V}'\left(t\right) = \mathbf{A}\mathbf{V}\left(t\right) - \mathbf{1}$$

Being the model time homogeneus, $\mathbf{V}'\left(t\right) = 0$ and we obtain the result $\mathbf{V} = \mathbf{A}^{-1} 1_n$.

2. Additional Material

This section contains some additional material excluded from the article.

First, Table 1 reports the moments from the SMM procedure, described in the appendix. Recall that our estimation procedure computes moments as $\varepsilon(t) = \left(e(t), \frac{\partial \hat{\mathcal{L}}}{\partial \Psi}\right)$, where $e(t)$ collects the differences between data observed financial quantities and their model's counterpart, which in turn depend on probabilities. The second term $\frac{\partial \hat{\mathcal{L}}}{\partial \Psi}$ are the scores of the likelihood function. Details of the procedure are in the appendix of the paper.

Second, Figure 1 reports a plot of inflation and earnings uncertainty computed from the model and proxies from the Survey of Professional Forecasters probabilities. In both cases, the uncertainty is computed as the conditional variance of the future inflation or economic growth. Details are in Section 4.1 of the paper.

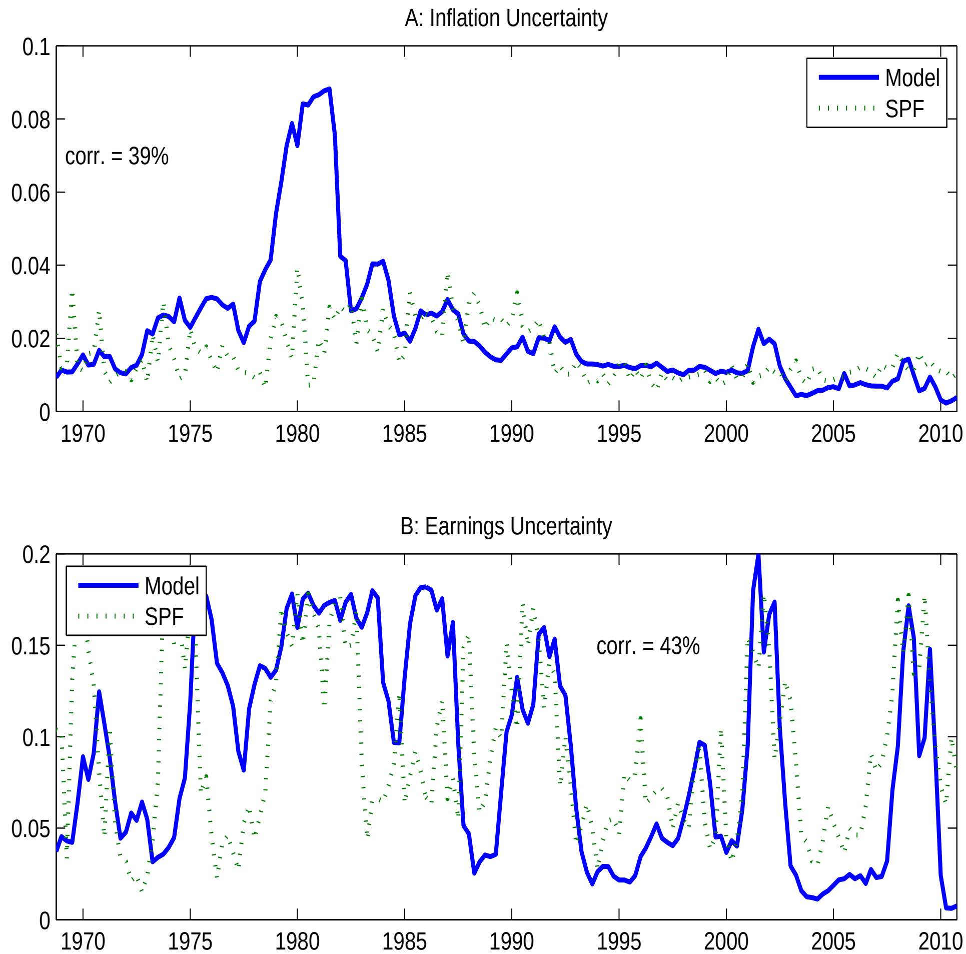
Third, Figure 2 plots the fitted asset prices from our model with six composite regimes, and compares them to the fitted asset prices from a simpler 2×2 model, that is, as model with two earnings drifts and two inflation regimes. As it is apparent, the 2×2 model, which has three state variables (beliefs), does not fit the asset pricing data well, especially on the dimension of bonds yields, volatility, and, most importantly, the covariance between stocks and bonds. The model is strongly rejected. Because the 4-regime model does not produce a dynamics of asset prices that is vaguely comparable to the data, we cannot hope to learn much from the model about the relation between fundamentals, volatilities, and price valuations. In contrast, the model with six composite regimes allow us to obtain numerous new predictions, that we can test in the data.

Table 1: Moments and Mean Absolute Errors from SMM

A: Pricing Errors		
Variable	Mean Error	MAE
P/E	0.4451	2.739
3-M Yield (%)	-0.866	1.47
5-Y Yield (%)	0.4008	1.257
S. Vol (%)	0.5131	5.238
1-Y B Vol (%)	0.3145	0.4659
5-y B Vol (%)	-0.5142	1.626
CS-5Y (%)	-0.0006	0.002704
CS-1Y (%)	-0.0002979	0.0008162
C1Y-5Y (%)	-0.002949	0.000427
Sharpe Ratio	-0.01257	0.01257
B: Scores of Likelihood Function		
Variable	Mean Error (Scaled)	MAE
β^1	-0.0006841	0.0393
β^2	-0.0002525	0.003492
β^3	-0.0001344	0.000706
β^4	1.28E-06	1.64E-05
θ^1	-0.0003834	0.0009879
θ^2	-0.0005433	0.00329
θ^3	0.0001237	8.44E-04
$\sigma_{Q,1}$	-0.03036	0.03059
$\sigma_{Q,2}$	-0.03768	0.03783
$\sigma_{E,2}$	-0.01143	0.01235
$\sigma_{S,1}$	-6.52E-05	4.58E-04
$\sigma_{S,2}$	-2.35E-05	4.13E-04
$\sigma_{C,2}$	0.01196	1.63E-02
$\sigma_{C,3}$	-0.02095	2.11E-02
σ_N	-0.0004874	5.29E-04
λ_{12}	1.58E-07	5.55E-06
λ_{13}	1.81E-07	7.446E-06
λ_{16}	9.27E-07	1.37E-05
λ_{21}	-2.85E-05	1.06E-04
λ_{23}	-2.03E-06	9.12E-05
λ_{24}	-1.69E-05	6.36E-05
λ_{25}	-1.65E-07	5.27E-06
λ_{26}	1.40E-07	3.07E-06
λ_{32}	-1.59E-05	2.09E-04
λ_{34}	-9.28E-07	3.53E-06
λ_{41}	-6.63E-06	6.92E-05
λ_{42}	4.51E-06	4.85E-05
λ_{51}	-2.85E-07	2.47E-06
λ_{53}	5.90E-08	1.61E-06
λ_{61}	-4.31E-05	7.80E-05

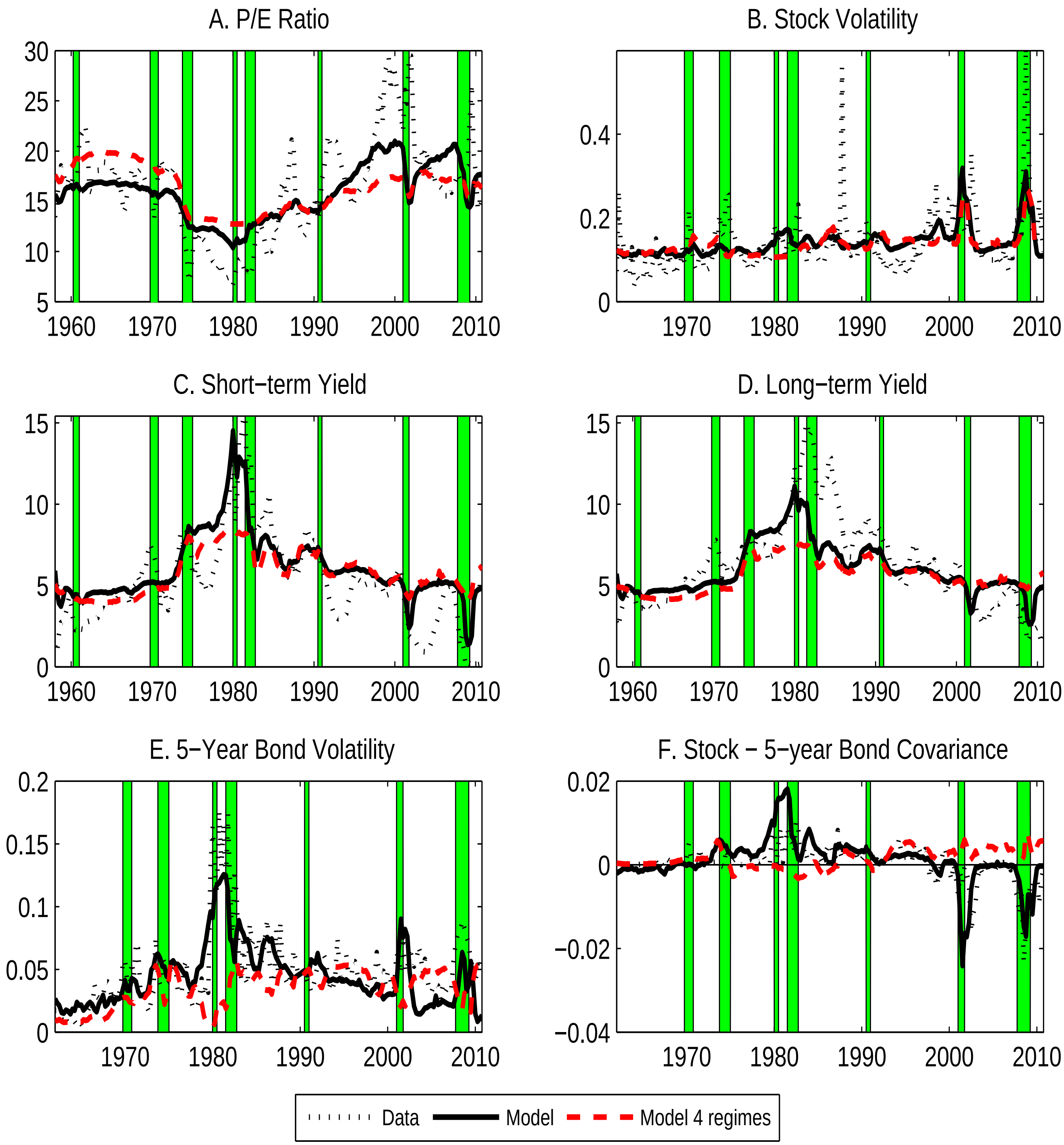
Notes: Panel A reports the pricing errors from the SMM procedure, while Panel B reports the moments from the scores of the likelihood function. The details of the estimation procedure are contained in the Appendix of the paper.

Figure 1: Model Uncertainty versus SPF Uncertainty



Notes: Panel A plots the conditional variance of future inflation computed using the model's probabilities (solid line) or the inflation probabilities extracted from the Survey of Professional Forecasters (dotted line). Panel B reports the conditional variance of future earnings growth using model's probabilities (solid line), and the conditional variance of economic growth using the Survey of Professional Forecasters' probabilities of real GDP decline. Details on the computation are in Section 4.1 of the paper.

Figure 2: Comparison with Four-Regime Model



Notes: Panel A - F compare the fitted asset prices from the six composite regimes presented in the main article to the restricted case with only four regimes. The four regimes have two inflation regimes, and two real earnings growth regimes.