Technical Appendix

to accompany

Uncertainty about Government Policy and Stock Prices

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Background Information.

Firm profitability evolves stochastically for all $t \in [0,T]$:

$$d\Pi_t^i = (\mu + g_t)\,dt + \sigma dZ_t + \sigma_1 dZ_t^i \ . \tag{B1}$$

The prior distributions of both g^{old} and g^{new} at time 0 are normal:

$$g \sim N\left(0, \sigma_g^2\right) \ . \tag{B2}$$

Learning.

Proposition 1. *Observing the continuum of signals $d\Pi_t^i$ in equation (B1) across all firms $i \in [0,1]$ is equivalent to observing a single aggregate signal about g_t:*

$$ds_t = (\mu + g_t)\,dt + \sigma dZ_t. \tag{B3}$$

Under the prior in equation (B2), the posterior for g_t at any time $t \in [0,T]$ is given by

$$g_t \sim N\left(\widehat{g}_t, \widehat{\sigma}_t^2\right). \tag{B4}$$

For all $t \le \tau$, the mean and the variance of this posterior distribution evolve as

$$d\widehat{g}_t = \widehat{\sigma}_t^2 \sigma^{-1} d\widehat{Z}_t \tag{B5}$$

$$\widehat{\sigma}_t^2 = \frac{1}{\frac{1}{\sigma_g^2} + \frac{1}{\sigma^2} t}, \tag{B6}$$

where the "expectation error" $d\widehat{Z}_t$ is given by $d\widehat{Z}_t = (ds_t - E_t(ds_t))/\sigma$ for all $t \in [0,T]$. If there is no policy change at time τ, then the processes (B5) and (B6) hold also for $t > \tau$. If there is a policy change at τ, then $\widehat{g}_t$ jumps from $\widehat{g}_\tau$ to zero right after the policy change, and for $t > \tau$, $\widehat{g}_t$ follows the process in equation (B5). In addition, for $t > \tau$, $\widehat{\sigma}_t$ follows

$$\widehat{\sigma}_t^2 = \frac{1}{\frac{1}{\sigma_g^2} + \frac{1}{\sigma^2}(t - \tau)}. \tag{B7}$$

Proof of Proposition 1: Let Δ denote a small time interval. Each signal $s_t^i \equiv d\Pi_t^i$ can be written as the sum of a common component and idiosyncratic noise. The common component is $c_t = (\mu + g_t)\Delta + \sigma\varepsilon_t$, where $\varepsilon_t \sim N(0,\Delta)$. Therefore, each signal at time t is given by

$$s_t^i = c_t + \sigma_1 \varepsilon_t^i$$

where the ε_t^i's are cross-sectionally independent and distributed as $\varepsilon_t^i \sim N(0,\Delta)$. Consider the information from the cross-section of signals for a given time t. Conditional on c_t, all firm-level signals s_t^i are independent, as a result of which they reveal c_t perfectly. However, these signals cannot reveal more than c_t for any t, because c_t is the same across signals. It follows that the agents' information set includes the common component c_t and nothing else. As $\Delta \to 0$, we have $c_t \to ds_t$. This common signal is equal to average profitability across firms: $ds_t = \int_0^1 d\Pi_t^i di$. Substituting for $d\Pi_t^i$ from (B1) yields

$$ds_t = (\mu + g_t)dt + \sigma dZ_t + \sigma_1 \int_0^1 dZ_t^i \ .$$

The Law of Large Numbers implies that the last integral is identically zero, as $\int_0^1 dZ_t^i = E^i\left[dZ_t^i\right] = 0$, where $E^i[\ ]$ denotes an expectation taken across i.

Having established the equivalence in (B3), the remainder of the proposition follows from standard results about the Kalman Bucy filter (see e.g. Liptser and Shiryaev (1977)).

QED (Proposition 1)

Optimal Changes in Government Policy.

Statement from the text: The aggregate capital at time T, $B_T = \int_0^1 B_T^i di$, is given by

$$B_T = B_\tau e^{\left(\mu + g - \frac{\sigma^2}{2}\right)(T-\tau) + \sigma(Z_T - Z_\tau)}, \tag{B8}$$

where $g \equiv g^{\text{old}}$ if there is no policy change and $g \equiv g^{\text{new}}$ if there is one.

Proof: From the capital evolution equation $dB_t^i = B_t^i d\Pi_t^i$, where $d\Pi_t^i$ is given in (B1), we immediately obtain the following expression for firm i's capital at time T:

$$B_T^i = B_\tau^i e^{\left(\mu + g - \frac{\sigma^2}{2} - \frac{\sigma_1^2}{2}\right)(T-\tau) + \sigma(Z_T - Z_\tau) + \sigma_1\left(Z_T^i - Z_\tau^i\right)}, \tag{B9}$$

where $g \equiv g^{\text{old}}$ if there is no policy change and $g \equiv g^{\text{new}}$ if there is one. Aggregating across firms, we obtain

$$B_T = \int_0^1 B_T^i di = e^{\left(\mu + g - \frac{\sigma^2}{2} - \frac{\sigma_1^2}{2}\right)(T-\tau) + \sigma(Z_T - Z_\tau)} \int_0^1 B_\tau^i e^{\sigma_1\left(Z_T^i - Z_\tau^i\right)} di \ . \tag{B10}$$

The Law of Large Numbers implies that

$$\int_0^1 B_\tau^i e^{\sigma_1\left(Z_T^i - Z_\tau^i\right)} di = E^i\left[B_\tau^i e^{\sigma_1\left(Z_T^i - Z_\tau^i\right)}\right] = E^i\left[B_\tau^i\right] E^i\left[e^{\sigma_1\left(Z_T^i - Z_\tau^i\right)}\right] \ , \tag{B11}$$

where the last step follows from the fact that the random variables B_τ^i and $(Z_T^i - Z_\tau^i)$ are independent of each other. The first expectation on the right-hand side of (B11) is

$$E^i\left[B_\tau^i\right] = \int_0^1 B_\tau^i di = B_\tau \ .$$

The second expectation is $E^i\left[e^{\sigma_1\left(Z_T^i - Z_\tau^i\right)}\right] = e^{\frac{1}{2}\sigma_1^2(T-\tau)}$. Substituting both expectations into (B10), we obtain the claim in (B8).

QED (Statement from the text)

Proposition 2. *The government changes its policy at time τ if and only if*

$$\widehat{g}_\tau \;<\; \underline{g}(c)\;, \tag{B12}$$

where

$$\underline{g}(c) \;=\; -\frac{\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)(\gamma-1)(T-\tau)}{2} \;-\; \frac{c}{(T-\tau)(\gamma-1)}\;. \tag{B13}$$

Proof of Proposition 2: Using the market clearing condition $W_T = B_T$, we can use (B8) to compute the expected utility at time T conditional on a policy change ("yes") or no policy change ("no"). The expectation is conditional on the government's information set, which includes the realization of the political cost c. Recall that if the government changes its policy, then $g \sim N(0, \sigma_g^2)$; if it does not, then $g \sim N(\widehat{g}_\tau, \widehat{\sigma}_\tau^2)$.

$$E\left[\frac{CW_T^{1-\gamma}}{1-\gamma}|yes\right] \;=\; \frac{B_\tau^{1-\gamma}}{1-\gamma}e^{c+(1-\gamma)(\mu-\frac{1}{2}\sigma^2)(T-\tau)+\frac{1}{2}\sigma_g^2(1-\gamma)^2(T-\tau)^2+\frac{(1-\gamma)^2}{2}\sigma^2(T-\tau)} \tag{B14}$$

$$E\left[\frac{W_T^{1-\gamma}}{1-\gamma}|no\right] \;=\; \frac{B_\tau^{1-\gamma}}{1-\gamma}e^{(1-\gamma)(\mu+\widehat{g}_\tau-\frac{1}{2}\sigma^2)(T-\tau)+\frac{1}{2}\widehat{\sigma}_\tau^2(1-\gamma)^2(T-\tau)^2+\frac{(1-\gamma)^2}{2}\sigma^2(T-\tau)} \tag{B15}$$

The claim of the proposition follows immediately from the optimality condition

$$E\left[\frac{CW_T^{1-\gamma}}{1-\gamma}|yes\right] > E\left[\frac{W_T^{1-\gamma}}{1-\gamma}|no\right]\;.$$

QED (Proposition 2)

Stock Prices.

Proposition 3. *Each firm's stock return at the announcement of a policy change is given by*

$$R\left(\widehat{g}_\tau\right) \;=\; \frac{\left(1-p\left(\widehat{g}_\tau\right)\right)F\left(\widehat{g}_\tau\right)\left(1-G\left(\widehat{g}_\tau\right)\right)}{p\left(\widehat{g}_\tau\right)+\left(1-p\left(\widehat{g}_\tau\right)\right)F\left(\widehat{g}_\tau\right)G\left(\widehat{g}_\tau\right)}\;, \tag{B16}$$

where

$$F\left(\widehat{g}_\tau\right) \;=\; e^{-\gamma\widehat{g}_\tau(T-\tau)-\frac{1}{2}\gamma^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)} \tag{B17}$$

$$G\left(\widehat{g}_\tau\right) \;=\; e^{\widehat{g}_\tau(T-\tau)-\frac{1}{2}(1-2\gamma)(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)} \tag{B18}$$

$$p\left(\widehat{g}_\tau\right) \;=\; N\left(\widehat{g}_\tau\left(1-\gamma\right)\left(T-\tau\right)-\frac{\left(1-\gamma\right)^2}{2}\left(T-\tau\right)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right);\; -\frac{\sigma_c^2}{2},\sigma_c^2\right)\;, \tag{B19}$$

and $N(x;a,b)$ denotes the c.d.f. of a normal distribution with mean a and variance b.

The proof of Proposition 3 comes after the proof of the following statement in the text.

Statement from the text: Right after time τ, at time $\tau+$, the market value of each firm i takes one of two values:

$$M^i_{\tau+} = \begin{cases} M^{i,yes}_{\tau+} = B^i_{\tau+}\; e^{\left(\mu-\gamma\sigma^2\right)(T-\tau)+\frac{1-2\gamma}{2}(T-\tau)^2\sigma_g^2} & \text{if policy changes} \\ M^{i,no}_{\tau+} = B^i_{\tau+}\; e^{\left(\mu-\gamma\sigma^2+\widehat{g}_\tau\right)(T-\tau)+\frac{1-2\gamma}{2}(T-\tau)^2\widehat{\sigma}^2_\tau} & \text{if policy does not change} \end{cases} \tag{B20}$$

Right before time τ, the market value of firm i is

$$M^i_\tau = \omega M^{i,yes}_{\tau+} + (1-\omega)\, M^{i,no}_{\tau+}, \tag{B21}$$

where the weight ω, which is always between 0 and 1, is given by

$$\omega = \frac{p_\tau}{p_\tau + (1-p_\tau)\, F\left(\widehat{g}_\tau\right)}\;, \tag{B22}$$

using the abbreviated notation $p_\tau \equiv p\left(\widehat{g}_\tau\right)$.

Proof: The stochastic discount factor is $\pi_t = \lambda^{-1} E_t\left[B_T^{-\gamma}\right]$. Its value right after the policy decision, at time $\tau+$, is given by

$$\pi_{\tau+} = \lambda^{-1} B^{-\gamma}_{\tau+} E_t\left[e^{-\gamma(\mu+g-\frac{1}{2}\sigma^2)(T-\tau)-\gamma\sigma(Z_T-Z_\tau)}\right] \tag{B23}$$

$$= \begin{cases} \pi^{yes}_{\tau+} = \lambda^{-1} B^{-\gamma}_{\tau+}\; e^{\left(-\gamma\mu+\frac{1}{2}\gamma(\gamma+1)\sigma^2\right)(T-\tau)+\frac{\gamma^2}{2}(T-\tau)^2\sigma_g^2} & \text{if policy changes} \\ \pi^{no}_{\tau+} = \lambda^{-1} B^{-\gamma}_{\tau+}\; e^{\left(-\gamma(\mu+\widehat{g}_\tau)+\frac{1}{2}\gamma(\gamma+1)\sigma^2\right)(T-\tau)+\frac{\gamma^2}{2}(T-\tau)^2\widehat{\sigma}^2_\tau} & \text{if policy does not change} \end{cases} \tag{B24}$$

where we have used (B8). Right before the policy decision, we have

$$\pi_\tau = E_\tau\left[\pi_{\tau+}\right] = p_\tau \pi^{yes}_{\tau+} + (1-p_\tau)\pi^{no}_{\tau+}, \tag{B25}$$

where $p_\tau \equiv p(\widehat{g}_\tau) = Prob\left(\widehat{g}_\tau < \underline{g}(c)\right)$ is the probability of a policy change from the perspective of investors (who know $\widehat{g}_\tau$ but not c). It is easy to see from (B13) that this probability is given by expression (B19).

The market value of stock i is given by $M^i_t = \pi_t^{-1} E_t\left[\pi_T B^i_T\right] = \pi_t^{-1}\lambda^{-1} E_t\left[B_T^{-\gamma} B^i_T\right]$. Right after a policy decision, at time $\tau+$, using both (B8) and (B9), we obtain

$$\begin{aligned} E_{\tau+}\left[B_T^{-\gamma}B^i_T\right] &= B^{-\gamma}_{\tau+}B^i_{\tau+}E_{\tau+}\left[e^{-\gamma(\mu+g-\frac{1}{2}\sigma^2)(T-\tau)-\gamma\sigma(Z_T-Z_\tau)}e^{\left(\mu+g-\frac{\sigma^2}{2}-\frac{\sigma_1^2}{2}\right)(T-\tau)+\sigma(Z_T-Z_\tau)+\sigma_1\left(Z^i_T-Z^i_\tau\right)}\right] \\ &= B^{-\gamma}_{\tau+}B^i_{\tau+}E_{\tau+}\left[e^{(1-\gamma)(\mu+g-\frac{1}{2}\sigma^2)(T-\tau)+(1-\gamma)\sigma(Z_T-Z_\tau)}e^{-\frac{1}{2}\sigma_1^2(T-\tau)+\sigma_1\left(Z^i_T-Z^i_\tau\right)}\right] \\ &= B^{-\gamma}_{\tau+}B^i_{\tau+}E_{\tau+}\left[e^{(1-\gamma)(\mu+g-\frac{1}{2}\sigma^2)(T-\tau)+(1-\gamma)\sigma(Z_T-Z_\tau)}\right]E_{\tau+}\left[e^{-\frac{1}{2}\sigma_1^2(T-\tau)+\sigma_1\left(Z^i_T-Z^i_\tau\right)}\right] \\ &= B^{-\gamma}_{\tau+}B^i_{\tau+}E_{\tau+}\left[e^{(1-\gamma)(\mu+g-\frac{1}{2}\sigma^2)(T-\tau)+(1-\gamma)\sigma(Z_T-Z_\tau)}\right] \\ &= \begin{cases} B^{-\gamma}_{\tau+}B^i_{\tau+}\; e^{\left((1-\gamma)\mu+\frac{1}{2}\gamma(\gamma-1)\sigma^2\right)(T-\tau)+\frac{(1-\gamma)^2}{2}(T-\tau)^2\sigma_g^2} & \text{if policy changes} \\ B^{-\gamma}_{\tau+}B^i_{\tau+}\; e^{\left((1-\gamma)(\mu+\widehat{g}_\tau)+\frac{1}{2}\gamma(\gamma-1)\sigma^2\right)(T-\tau)+\frac{(1-\gamma)^2}{2}(T-\tau)^2\widehat{\sigma}^2_\tau} & \text{if policy does not change} \end{cases} \end{aligned}$$

These expressions can be substituted into $M_{\tau+}^{i,yes} = \lambda^{-1}E_{\tau+}\left[B_T^{-\gamma}B_T^i|yes\right]/\pi_{\tau+}^{yes}$ and $M_{\tau+}^{i,no} = \lambda^{-1}E_{\tau+}\left[B_T^{-\gamma}B_T^i|no\right]/\pi_{\tau+}^{no}$ to yield equation (B20).

Finally, the stock price right before the policy decision announcement is equal to

$$\begin{aligned} M_\tau^i &= \frac{E_\tau\left[E_{\tau+}\left[\lambda^{-1}B_T^{-\gamma}B_T^i\right]\right]}{\pi_\tau} = \frac{p_\tau E_{\tau+}\left[\lambda^{-1}B_T^{-\gamma}B_T^i|yes\right] + (1-p_\tau)E_{\tau+}\left[\lambda^{-1}B_T^{-\gamma}B_T^i|no\right]}{p_\tau\pi_{\tau+}^{yes} + (1-p_\tau)\pi_{\tau+}^{no}} \\ &= \frac{p_\tau\pi_{\tau+}^{yes}M_{\tau+}^{i,yes} + (1-p_\tau)\pi_{\tau+}^{no}M_{\tau+}^{i,no}}{p_\tau\pi_{\tau+}^{yes} + (1-p_\tau)\pi_{\tau+}^{no}} \end{aligned}$$

which is equivalent to (B21) when we define

$$\omega = \frac{p_\tau\pi_{\tau+}^{yes}}{p_\tau\pi_{\tau+}^{yes} + (1-p_\tau)\pi_{\tau+}^{no}} = \frac{p_\tau}{p_\tau + (1-p_\tau)\frac{\pi_{\tau+}^{no}}{\pi_{\tau+}^{yes}}}$$

It follows immediately from (B24) that $F(\widehat{g}_\tau) = \frac{\pi_{\tau+}^{no}}{\pi_{\tau+}^{yes}}$.

QED (Statement from the text)

Proof of Proposition 3: From the definition of the announcement return, we have

$$R(\widehat{g}_\tau) = \frac{M_{\tau+}^{i,yes} - M_\tau^i}{M_\tau^i} = \frac{(1-\omega)(M_{\tau+}^{i,yes} - M_{\tau+}^{i,no})}{\omega M_{\tau+}^{i,yes} + (1-\omega)M_{\tau+}^{i,no}} = \frac{(1-\omega)(1 - M_{\tau+}^{i,no}/M_{\tau+}^{i,yes})}{\omega + (1-\omega)(M_{\tau+}^{i,no}/M_{\tau+}^{i,yes})} \quad \text{(B26)}$$

It is easy to see from (B20) that $G(\widehat{g}_\tau) = M_{\tau+}^{i,no}/M_{\tau+}^{i,yes}$. Substituting this expression and ω from (B22) proves the claim.

QED (Proposition 3)

Corollary 1. *As risk aversion $\gamma \to \infty$, the announcement return $R(\widehat{g}_\tau) \to -1$ for any $\widehat{g}_\tau$.*

Proof of Corollary 1: For any given $\widehat{g}_\tau$, expression (B19) implies that $p_\tau \to 0$ as $\gamma \to \infty$, as the term involving γ^2 dominates the limit. In addition, an application of l'Hospital's rule shows that $p_\tau/(F(\widehat{g}_\tau)G(\widehat{g}_\tau)) \to 0$. Finally, $G(\widehat{g}_\tau)$ trivially diverges to infinity. Therefore,

$$R(\widehat{g}_\tau) = \frac{(1-p_\tau)F(\widehat{g}_\tau)(1-G(\widehat{g}_\tau))}{p_\tau + (1-p_\tau)F(\widehat{g}_\tau)G(\widehat{g}_\tau)} = \frac{(1-p_\tau)(1/G(\widehat{g}_\tau) - 1)}{p_\tau/(F(\widehat{g}_\tau)G(\widehat{g}_\tau)) + (1-p_\tau)} \to -1 .$$

QED (Corollary 1)

Corollary 2. *As risk aversion $\gamma \to 1$, the expected value of the announcement return goes to zero* ($\mathrm{E}\{R(\widehat{g}_\tau)\} \to 0$)*, where the expectation is computed with respect to $\widehat{g}_\tau$ as of time 0.*

Proof of Corollary 2: From expression (B19), we have $p_\tau \to \overline{p} = \mathcal{N}(0, -\frac{\sigma_c^2}{2}, \sigma_c^2)$ as $\gamma \to 1$. In addition, $F(\widehat{g}_\tau) \to 1/G(\widehat{g}_\tau)$. Therefore,

$$\begin{aligned} R(\widehat{g}_\tau) &= \frac{(1-p_\tau)F(\widehat{g}_\tau)(1-G(\widehat{g}_\tau))}{p_\tau + (1-p_\tau)F(\widehat{g}_\tau)G(\widehat{g}_\tau)} = \frac{(1-p_\tau)(F(\widehat{g}_\tau) - F(\widehat{g}_\tau)G(\widehat{g}_\tau))}{p_\tau + (1-p_\tau)F(\widehat{g}_\tau)G(\widehat{g}_\tau)} \\ &\to (1-\overline{p})\left(e^{-\widehat{g}_\tau(T-\tau) - \frac{1}{2}(T-\tau)^2\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)} - 1\right) \end{aligned}$$

Recalling that $\widehat{g}_\tau \sim N(0, \sigma_g^2 - \widehat{\sigma}_\tau^2)$, it is easy to compute the expectation:

$$E\left[R\left(\widehat{g}_\tau\right)\right] \;=\; (1-\overline{p})\left(e^{+\frac{1}{2}(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)-\frac{1}{2}(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}-1\right)=0\;.$$

QED (Corollary 2)

Proposition 4. *The market value of each firm drops at the announcement of a policy change (i.e., $R\left(\widehat{g}_\tau\right) < 0$) if and only if*

$$\widehat{g}_\tau \;>\; g^*\;, \tag{B27}$$

where

$$g^* \;=\; -\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)(T-\tau)\left(\gamma-\frac{1}{2}\right). \tag{B28}$$

Proof of Proposition 4: From expression (B16), we see that $R(\widehat{g}_\tau) < 0$ if and only if $G(\widehat{g}_\tau) > 1$. From the formula for $G(\widehat{g}_\tau)$ in (B18), we see that the latter condition is satisfied if and only if condition (B27) is.

QED (Proposition 4)

Proposition 5. *The expected value of the announcement return conditional on a policy change is negative: $E\left[R(\widehat{g}_\tau)|Policy\ Change\right] < 0$.*

Proof of Proposition 5: The proof proceeds through four Lemmas, A1–A4.

Lemma A1: The announcement return is given by

$$R\left(x\right)=\frac{\left(1-N_c\left(x\right)\right)e^x}{N_c\left(x\right)+\left(1-N_c\left(x\right)\right)e^x}\left(e^{\frac{x}{\gamma-1}-\frac{1}{2}\gamma(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}-1\right)\;, \tag{B29}$$

where $N_c\left(x\right) = N\left(x, -\frac{\sigma_c^2}{2}, \sigma_c^2\right) = \Pr\left(c < x\right)$ is the cumulative normal density, and the random variable x has the normal distribution

$$x \sim N\left(-\frac{\sigma_x^2}{2}, \sigma_x^2\right)\;,$$

where

$$\sigma_x^2 = \left(1-\gamma\right)^2\left(T-\tau\right)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)\;. \tag{B30}$$

Moreover, a policy change occurs if and only if the political cost is sufficiently low, i.e.

$$c < x\;.$$

Proof of Lemma A1: We can rewrite the formula for $R(\widehat{g}_\tau)$ in Proposition 3 as

$$\begin{aligned} R(\widehat{g}_\tau) &= \frac{(1-p(\widehat{g}_\tau))\,F(\widehat{g}_\tau)\,G(\widehat{g}_\tau)\,(G^{-1}(\widehat{g}_\tau)-1)}{p(\widehat{g}_\tau)+(1-p(\widehat{g}_\tau))\,F(\widehat{g}_\tau)\,G(\widehat{g}_\tau)} \\ &= \frac{(1-p(\widehat{g}_\tau))\,V(\widehat{g}_\tau)\,(G^{-1}(\widehat{g}_\tau)-1)}{p(\widehat{g}_\tau)+(1-p(\widehat{g}_\tau))\,V(\widehat{g}_\tau)} \end{aligned}$$

where

$$\begin{aligned} V(\widehat{g}_\tau) &= F(\widehat{g}_\tau)\,G(\widehat{g}_\tau) = e^{-\gamma\widehat{g}_\tau(T-\tau)-\frac{1}{2}\gamma^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}e^{\widehat{g}_\tau(T-\tau)-\frac{1}{2}(1-2\gamma)(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)} \\ &= e^{\widehat{g}_\tau(1-\gamma)(T-\tau)-\frac{1}{2}(1-\gamma)^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)} \end{aligned}$$

Note that the exponent in $V(\widehat{g}_\tau)$ is identical to the argument in $p(\widehat{g}_t)$ in (B19). Denoting this exponent by x,

$$x \equiv \widehat{g}_\tau(1-\gamma)(T-\tau) - \frac{1}{2}(1-\gamma)^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right) \;,$$

we can rewrite the announcement return as (with a slight abuse of notation):

$$R(\widehat{g}_\tau) = R(x) = \frac{\left(1-N\left(x,-\frac{\sigma_c^2}{2},\sigma_c^2\right)\right)e^x\left(G^{-1}(\widehat{g}_\tau)-1\right)}{N\left(x,-\frac{\sigma_c^2}{2},\sigma_c^2\right)+\left(1-N\left(x,-\frac{\sigma_c^2}{2},\sigma_c^2\right)\right)e^x}$$

In addition, expressing $\widehat{g}_\tau$ in terms of x,

$$\widehat{g}_\tau = \frac{x}{(1-\gamma)(T-\tau)} + \frac{1}{2}(1-\gamma)(T-\tau)\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)$$

we also obtain

$$\begin{aligned} G^{-1}(\widehat{g}_\tau) &= e^{-\widehat{g}_\tau(T-\tau)+\frac{1}{2}(1-2\gamma)(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)} \\ &= e^{-\frac{x}{(1-\gamma)}-\frac{1}{2}(1-\gamma)(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)+\frac{1}{2}(1-2\gamma)(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)} \\ &= e^{\frac{x}{\gamma-1}-\frac{1}{2}\gamma(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)} \end{aligned}$$

which leads to (B29).

From its definition above, x is normally distributed as of time 0, $x \sim N(\mu_x, \sigma_x^2)$, where

$$\begin{aligned} \sigma_x^2 &= (1-\gamma)^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right) \\ \mu_x &= -\frac{1}{2}(1-\gamma)^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right) = -\frac{\sigma_x^2}{2}\;. \end{aligned}$$

Finally, the condition for a policy change can also be expressed in terms of x. Recall that a policy change occurs if and only if

$$\widehat{g}_\tau < \underline{g}(c) = -\frac{\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)(\gamma-1)(T-\tau)}{2} - \frac{c}{(T-\tau)(\gamma-1)}\;.$$

This inequality is equivalent to

$$\begin{aligned} c &< -\widehat{g}_{\tau}(T-\tau)(\gamma-1) - \frac{\left(\sigma_g^2 - \widehat{\sigma}_{\tau}^2\right)(\gamma-1)^2 (T-\tau)^2}{2} \\ &= x \ . \end{aligned}$$

Q.E.D. (Lemma A1)

Lemma A2: The Expected Annoucement Return (EAR) can be written as

$$E\left[R(x) \,|\text{Policy Change}\right] = \frac{1}{\int \phi_x(x) N_c(x)\, dx} \times \int k(x) \left(e^{\frac{x}{\gamma-1} - \frac{1}{2}\gamma(T-\tau)^2\left(\sigma_g^2 - \widehat{\sigma}_{\tau}^2\right)} - 1 \right) \phi_x(x)\, dx \ ,$$

where $\phi_x(x)$ denotes the normal probability density function of x and

$$k(x) = \frac{(1 - N_c(x)) N_c(x) e^x}{N_c(x) + (1 - N_c(x)) e^x} \ .$$

Proof of Lemma A2: We can write

$$E\left[R(x) \,|\text{Policy Change}\right] = \int R(x)\, \phi_x(x|\text{Policy Change})\, dx \ ,$$

where $\phi_x(x|\text{Policy Change})$ denotes the density of x conditional on a policy change at τ. We know that a policy change occurs iff $x > c$, where c has a normal density that is independent of x. That is,

$$\begin{aligned} \phi_x(x|\text{Policy Change}) &= \int \phi_x(x|yes, c)\, \phi_c(c)\, dc = \int \phi_x(x|x > c, c)\, \phi_c(c)\, dc \\ &\propto \int \phi_x(x)\, 1_{\{x>c\}} \phi_c(c)\, dc = \phi_x(x) \int_{-\infty}^{x} \phi_c(c)\, dc \\ &= \phi_x(x) N_c(x) \ . \end{aligned}$$

Dividing by the integration constant, we obtain

$$\phi_x(x|\text{Policy Change}) = \frac{\phi_x(x) N_c(x)}{\int \phi_x(x) N_c(x)\, dx} \ .$$

Therefore, we can substitute

$$\begin{aligned} E\left[R(x) \,|\text{Policy Change}\right] &= \int R(x)\, \phi_x(x|\text{Policy Change})\, dx \\ &= \frac{1}{\int \phi_x(x) N_c(x)\, dx} \times \\ &\quad \times \int \frac{(1 - N_c(x)) e^x}{N_c(x) + (1 - N_c(x)) e^x} \left(e^{\frac{x}{\gamma-1} - \frac{1}{2}\gamma(T-\tau)^2\left(\sigma_g^2 - \widehat{\sigma}_{\tau}^2\right)} - 1 \right) \phi_x(x) N_c(x)\, dx \\ &= \frac{1}{\int \phi_x(x) N_c(x)\, dx} \times \\ &\quad \times \int \frac{(1 - N_c(x)) N_c(x) e^x}{N_c(x) + (1 - N_c(x)) e^x} \left(e^{\frac{x}{\gamma-1} - \frac{1}{2}\gamma(T-\tau)^2\left(\sigma_g^2 - \widehat{\sigma}_{\tau}^2\right)} - 1 \right) \phi_x(x)\, dx \ . \end{aligned}$$

Defining $k(x)$ as in the claim yields the lemma.

Q.E.D. (Lemma A2)

The denominator in $E[R(x)|\text{Policy Change}]$, namely $\int \phi_x(x) N_c(x)\, dx$, is always positive. Therefore, we only need to consider the numerator, which we denote as

$$S = \int k(x) \left(e^{\frac{x}{\gamma-1} - \frac{1}{2}\gamma(T-\tau)^2 \left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)} - 1 \right) \phi_x(x)\, dx \tag{B31}$$

It is convenient to express S as a function of σ_x and a constant ℓ. In particular, from the definition of σ_x in (B30), we obtain

$$\gamma - 1 = \frac{\sigma_x}{(T-\tau)\sqrt{\sigma_g^2 - \widehat{\sigma}_\tau^2}} \ .$$

Substituting for γ in (B31), we obtain the equivalent expression

$$S = \int k(x) \left(e^{\left(\frac{x - \frac{1}{2}\sigma_x^2}{\sigma_x}\right)\ell - \frac{1}{2}\ell^2} - 1 \right) \phi_x(x)\, dx \ , \tag{B32}$$

where

$$\ell = (T-\tau)\sqrt{\sigma_g^2 - \widehat{\sigma}_\tau^2} \ .$$

We now rewrite S in an equivalent form, swapping the exponential term for the function k evaluated at a different value.

Lemma A3: An equivalent expression for S is

$$S = E\left[e^{-m} k(x+m)\right] - E[k(x)] \ ,$$

where $x \sim N\left(-\frac{1}{2}\sigma_x^2, \sigma_x^2\right)$ and

$$m = \sigma_x \ell \ .$$

Proof of Lemma A3: Rewrite

$$S = \int k(x) \left(e^{\left(\frac{x - \frac{1}{2}\sigma_x^2}{\sigma_x}\right)\ell - \frac{1}{2}\ell^2} - 1 \right) \phi_x(x)\, dx = e^{-\frac{1}{2}\ell^2} E\left[k(x)\, e^{\left(\frac{x - \frac{1}{2}\sigma_x^2}{\sigma_x}\right)\ell} \right] - E[k(x)]$$

We now rewrite the first expectation as follows

$$E\left[k(x)\, e^{\left(\frac{x - \frac{1}{2}\sigma_x^2}{\sigma_x}\right)\ell} \right] = \int k(x)\, e^{\left(\frac{x - \frac{1}{2}\sigma_x^2}{\sigma_x}\right)\ell} \frac{e^{-\frac{1}{2}\frac{\left(x + \frac{1}{2}\sigma_x^2\right)^2}{\sigma_x^2}}}{\sqrt{2\pi}\sigma_x} dx \ .$$

The product of the exponentials can be written as

$$e^{\left(\frac{x-\frac{1}{2}\sigma_x^2}{\sigma_x}\right)\ell}\frac{e^{\frac{-\left(x+\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x}=\frac{e^{\frac{-\left(x+\frac{1}{2}\sigma_x^2\right)^2+\left(x-\frac{1}{2}\sigma_x^2\right)2\ell\sigma_x}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x}$$

Now, we can rewrite the exponent in the exponential function as follows

$$\begin{aligned}
Exponent &= \frac{-\left(x+\frac{1}{2}\sigma_x^2\right)^2+\left(x-\frac{1}{2}\sigma_x^2\right)2\ell\sigma_x}{2\sigma_x^2}\\
&= \frac{-\left(x^2+\left(\frac{1}{2}\sigma_x^2\right)^2+x\sigma_x^2\right)+\left(x-\frac{1}{2}\sigma_x^2\right)2\ell\sigma_x}{2\sigma_x^2}\\
&= \frac{-\left(x^2+\left(\frac{1}{2}\sigma_x^2\right)^2+x\sigma_x^2-x2\ell\sigma_x\right)-\sigma_x^2\ell\sigma_x}{2\sigma_x^2}\\
&= \frac{-\left(x^2+\left(\frac{1}{2}\sigma_x^2\right)^2+x\left(\sigma_x^2-2\ell\sigma_x\right)\right)-\sigma_x^3\ell}{2\sigma_x^2}\\
&= \frac{-\left(x+\frac{1}{2}\left(\sigma_x^2-2\ell\sigma_x\right)\right)^2+\left(\left(\ell\sigma_x\right)^2-\sigma_x^3\ell\right)-\sigma_x^3\ell}{2\sigma_x^2}\\
&= \frac{-\left(x+\frac{1}{2}\left(\sigma_x^2-2\ell\sigma_x\right)\right)^2+\left(\ell\sigma_x\right)^2-2\sigma_x^3\ell}{2\sigma_x^2}\\
&= \frac{-\left(x+\frac{1}{2}\left(\sigma_x^2-2\ell\sigma_x\right)\right)^2}{2\sigma_x^2}+\frac{1}{2}\ell^2-\sigma_x\ell\ .
\end{aligned}$$

Thus, we obtain the identity

$$\begin{aligned}
e^{\left(\frac{x-\frac{1}{2}\sigma_x^2}{\sigma_x}\right)\ell}\frac{e^{\frac{-\left(x+\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x} &= \frac{e^{\frac{-\left(x+\frac{1}{2}\left(\sigma_x^2-2\ell\sigma_x\right)\right)^2}{2\sigma_x^2}+\frac{1}{2}\ell^2-\sigma_x\ell}}{\sqrt{2\pi}\sigma_x}\\
&= e^{+\frac{1}{2}\ell^2-\sigma_x\ell}\left[\frac{e^{\frac{-\left(x+\frac{1}{2}\left(\sigma_x^2-2\ell\sigma_x\right)\right)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x}\right]
\end{aligned}$$

Note that the expression in the parenthesis is the density of a normal with mean $-\frac{1}{2}\left(\sigma_x^2-2\ell\sigma_x\right)=-\frac{1}{2}\sigma_x^2+\ell\sigma_x$ and variance σ_x^2. Substituting in the expression for S, we obtain

$$\begin{aligned}
S &= e^{-\frac{1}{2}\ell^2}E\left[k\left(x\right)e^{\left(\frac{x-\frac{1}{2}\sigma_x^2}{\sigma_x}\right)\ell}\right]-E\left[k\left(x\right)\right]\\
&= e^{-\frac{1}{2}\ell^2}\int k\left(x\right)e^{\left(\frac{x-\frac{1}{2}\sigma_x^2}{\sigma_x}\right)\ell}\frac{e^{-\frac{1}{2}\frac{\left(x+\frac{1}{2}\sigma_x^2\right)^2}{\sigma_x^2}}}{\sqrt{2\pi}\sigma_x}dx-E\left[k\left(x\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= e^{-\frac{1}{2}\ell^2}\int k(x)\, e^{+\frac{1}{2}\ell^2-\sigma_x\ell}\frac{e^{\frac{-\left(x+\frac{1}{2}\left(\sigma_x^2-2\ell\sigma_x\right)\right)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x}dx - E[k(x)] \\
&= e^{-\frac{1}{2}\ell^2}E\left[k(\widehat{x})\, e^{\frac{1}{2}\ell^2-\sigma_x\ell}\right] - E[k(x)] \\
&= e^{-\sigma_x\ell}E[k(\widehat{x})] - E[k(x)] \ ,
\end{aligned}
$$

where

$$
\widehat{x} \sim N\left(-\frac{1}{2}\sigma_x^2+\ell\sigma_x, \sigma_x^2\right)
$$

Because x and $\widehat{x}$ differ from each other only because of the mean $m = \ell\sigma_x$, we have

$$
\widehat{x} = x + m
$$

(formally, we have defined $\widehat{x}$ from x in the derivation above). This implies that we can rewrite S as

$$
S = E\left[e^{-m}k(x+m)\right] - E[k(x)] \ .
$$

Q.E.D. (Lemma A3)

The following lemma yields one last transformation of S.

Lemma A4: S can be equivalently expressed as

$$
S = E\left[\overline{k}(x+m) - \overline{k}(x)\right] \ , \tag{B33}
$$

where

$$
x \sim N\left(\frac{1}{2}\sigma_x^2, \sigma_x^2\right)
$$

and

$$
\overline{k}(x) = \frac{(1-N_c(x))\, N_c(x)}{N_c(x) + (1-N_c(x))\, e^x} \ . \tag{B34}
$$

(Note that we have redefined x so the mean of x changes the sign compared to the x defined in Lemma A1.)

Proof of Lemma A4: From Lemma A3, we have

$$
S = E\left[e^{-m}k(x+m)\right] - E[k(x)] \ .
$$

Substituting for $k(x+m)$ and $k(x)$,

$$
\begin{aligned}
S &= E\left[e^{-m}\frac{(1-N_c(x+m))\, N_c(x+m)\, e^{x+m}}{N_c(x+m)+(1-N_c(x+m))\, e^{x+m}}\right] - E\left[\frac{(1-N_c(x))\, N_c(x)\, e^x}{N_c(x)+(1-N_c(x))\, e^x}\right] \\
&= E\left[\frac{(1-N_c(x+m))\, N_c(x+m)\, e^{x}}{N_c(x+m)+(1-N_c(x+m))\, e^{x+m}}\right] - E\left[\frac{(1-N_c(x))\, N_c(x)\, e^x}{N_c(x)+(1-N_c(x))\, e^x}\right] \\
&= E\left[\overline{k}(x+m)\, e^x\right] - E\left[\overline{k}(x)\, e^x\right] = E\left[\left[\overline{k}(x+m) - \overline{k}(x)\right] e^x\right] \\
&= \int \left[\overline{k}(x+m) - \overline{k}(x)\right] e^x \frac{e^{-\frac{\left(x+\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x}dx \ ,
\end{aligned}
$$

where x is still defined as $x \sim N\left(-\frac{1}{2}\sigma_x^2, \sigma_x^2\right)$. We now carry out a transformation to swap the exponential e^x for a change in the mean of the distribution of x. The product of the exponentials is

$$\begin{aligned} e^x \frac{e^{-\frac{\left(x+\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x} &= \frac{e^{\frac{-\left(x+\frac{1}{2}\sigma_x^2\right)^2+x2\sigma_x^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x} = \frac{e^{\frac{-\left(x^2+\left(\frac{1}{2}\sigma_x^2\right)^2+x\sigma_x^2\right)+x2\sigma_x^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x} \\ &= \frac{e^{\frac{-\left(x^2+\left(\frac{1}{2}\sigma_x^2\right)^2-x\sigma_x^2\right)}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x} = \frac{e^{\frac{-\left(x-\left(\frac{1}{2}\sigma_x^2\right)\right)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x} , \end{aligned}$$

which is the density of the normal distribution with mean $\frac{1}{2}\sigma_x^2$ and variance σ_x^2 (note the change in the sign of the mean compared to the definition of x in Lemma A1). Substituting this identity, we obtain

$$\begin{aligned} S &= \int \left[\overline{k}(x+m) - \overline{k}(x)\right] e^x \frac{e^{-\frac{\left(x+\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x} dx = \int \left[\overline{k}(x+m) - \overline{k}(x)\right] \frac{e^{\frac{-\left(x-\left(\frac{1}{2}\sigma_x^2\right)\right)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x} dx \\ &= E\left[\overline{k}(x+m) - \overline{k}(x)\right] , \end{aligned}$$

where x is now redefined as

$$x \sim N\left(\frac{1}{2}\sigma_x^2, \sigma_x^2\right) .$$

Q.E.D. (Lemma A4)

To summarize Lemmas A1 through A4, we have proved that $E\left[R(x)|\text{Policy Change}\right] < 0$ if and only if $S < 0$, where S is given in (B33). To prove that $S < 0$, we now prove that the function $E\left[\overline{k}(x+m)\right]$ is decreasing in m for all $m > 0$.

First, we document some properties of the $\overline{k}(x)$ function, which is defined in (B34). This function is always positive, and it converges to zero for both $x \to -\infty$ and $x \to \infty$. In other words, the function $\overline{k}(x)$ is "hump-shaped". Second, we prove that $\overline{k}(x)$ peaks below zero; that is, $\overline{k}(x)$ is monotonically decreasing for all $x > 0$. To show this result, we show that the first derivative of $\overline{k}$ is negative for all $x > 0$. We have $\frac{d\overline{k}}{dx} < 0$ if and only if

$$-N_c'(x) N_c(x)^2 + (1 - N_c(x))^2 (N_c'(x) - N_c(x)) e^x < 0 ,$$

or, equivalently, if and only if

$$\left(1 - \frac{N_c(x)}{N_c'(x)}\right) < e^{-x} \left(\frac{N_c(x)}{1 - N_c(x)}\right)^2 .$$

Direct computation shows that this condition is always satisfied for any $x > 0$ and any σ_c. Third, we note that if the function $\overline{k}(x)$ were symmetric, the claim $S < 0$ would follow immediately. This is because the expectation $E\left[\overline{k}(x)\right]$ is computed with respect to the normal density of x whose mean $\mu_x = \frac{1}{2}\sigma_x^2$ is positive. Since the function $\overline{k}(x)$ peaks below zero, increasing the mean of x from $\mu_x > 0$ to $\mu_x + m$ would decrease the expected value of a symmetric $\overline{k}(x)$ for all $m > 0$. The function $\overline{k}(x)$ is almost symmetric: the numerator of $\overline{k}(x)$ in (B34) is symmetric about $-1/2\sigma_c^2$, while the denominator is given by $Den(x) = N_c(x) + (1 - N_c(x))f(x)$, where $f(x) = e^x$. If we had $f(x) = 1$ for all x, then it would be the case that $Den(x) = 1$ and $\overline{k}(x)$ would be symmetric. Instead, $f(x) = e^x > 1$ for all $x > 0$, which implies $Den(x) > 1$, so that $\overline{k}(x)$ is lower than the corresponding symmetric value. For $x < 0$, we have $f(x) = e^x < 1$, which implies $Den(x) < 1$ and thus $\overline{k}(x)$ is above the corresponding symmetric value. This characterization implies that $\overline{k}(x)$ is negatively skewed—the right tail of $\overline{k}(x)$ for $x > 0$ is thinner than it would be under the symmetric distribution. Since increasing the mean of x from μ_x to $\mu_x + m$ shifts the whole density to the right, the thinner right tail of $\overline{k}(x)$ reinforces the argument for $S < 0$ under symmetry. We conclude that $E[\overline{k}(x + m)] < E[\overline{k}(x)]$ for every $m > 0$.

QED (Proposition 5)

Proposition A1. In the benchmark model for $t \leq \tau$, the state price density is given by

$$\pi_t = B_t^{-\gamma}\Omega\left(\widehat{g}_t, t\right) \ , \tag{B35}$$

where

$$\begin{aligned}
\Omega\left(\widehat{g}_t, t\right) &= p_t^{yes} G_t^{yes} + \left(1 - p_t^{no}\right) G_t^{no} \\
G_t^{yes} &= e^{-\gamma\mu(T-t) - \gamma\widehat{g}_t(\tau - t) + \frac{\gamma^2}{2}\left((T-\tau)^2\widehat{\sigma}_g^2 + (\tau-t)^2\widehat{\sigma}_t^2\right) + \gamma(1+\gamma)\frac{\sigma^2}{2}(T-t)} \\
G_t^{no} &= e^{-\gamma(\mu+\widehat{g}_t)(T-t) + \frac{\gamma^2}{2}(T-t)^2\widehat{\sigma}_t^2 + \gamma(1+\gamma)\frac{\sigma^2}{2}(T-t)}
\end{aligned}$$

and

$$\begin{aligned}
p_t^{yes} &= N\left(\underline{g}(0); \widehat{g}_t - \gamma\widehat{\sigma}_t^2\left(\tau - t\right) + \frac{\sigma_c^2/2}{\left(T-\tau\right)\left(1-\gamma\right)}, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2 + \frac{\sigma_c^2}{\left(T-\tau\right)^2\left(1-\gamma\right)^2}\right) \\
p_t^{no} &= N\left(\underline{g}(0); \widehat{g}_t - \gamma\left[\widehat{\sigma}_t^2\left(T - t\right) - \left(T-\tau\right)\widehat{\sigma}_\tau^2\right] + \frac{\sigma_c^2/2}{\left(T-\tau\right)\left(1-\gamma\right)}, \widehat{\sigma}_t^2 - \sigma_\tau^2 + \frac{\sigma_c^2}{\left(T-\tau\right)^2\left(1-\gamma\right)^2}\right)
\end{aligned}$$

Proof of Proposition A1: We have

$$\pi_t = E_t\left[\pi_{\tau+}\right] = \int E_t\left[\pi_{\tau+}|c\right]\phi_c\left(c\right)dc \ ,$$

where $\phi_c(c)$ is the density of a normal distribution with mean $-\frac{1}{2}\sigma_c^2$ and variance σ_c^2. Consider

$$E_t\left[\pi_{\tau+}|c\right] = p_t\left(c\right)E_t\left[\pi_{\tau+}|c, \widehat{g}_\tau < \underline{g}\left(c\right)\right] + \left(1 - p_t\left(c\right)\right)E_t\left[\pi_{\tau+}|c, \widehat{g}_\tau > \underline{g}\left(c\right)\right] \ ,$$

where the cutoff $\underline{g}\left(c\right)$ is given in (B13), and

$$p_t\left(c\right) = \Pr\left(\widehat{g}_\tau < \underline{g}\left(c\right)\right) = \mathcal{N}\left(\underline{g}\left(c\right), \widehat{g}_t, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right) \ .$$

We proceed in two steps. In the first step, we compute the inner expectation conditional on a particular political cost c. In the second step, we integrate the cost out.

STEP 1: We compute the two conditional expectations separately, first computing $E_t\left[\pi_{\tau+}|c,\widehat{g}_\tau<\underline{g}(c)\right]$ and then $E_t\left[\pi_{\tau+}|c,\widehat{g}_\tau>\underline{g}(c)\right]$.

CASE 1. POLICY CHANGE: The state price density at time τ right after a policy change (i.e., at time $\tau+$) is given in (B24) as

$$\pi_{\tau+}=\lambda^{-1}B_{\tau+}^{-\gamma}e^{-\gamma\left(\mu-\frac{\sigma^2}{2}\right)(T-\tau)+\frac{\gamma^2}{2}\left((T-\tau)^2\sigma_g^2+\sigma^2(T-\tau)\right)}\ .$$

In what follows, we omit the constant λ^{-1} which drops out later, and we denote $B_{\tau+}$ simply as B_τ (this is innocuous because $B_\tau=B_{\tau+}$ by continuity; there are no jumps in capital at time τ). Consider the conditional expectation

$$\begin{aligned}E_t\left[\pi_{\tau+}|\widehat{g}_\tau<\underline{g}(c),c\right] &= E_t\left[B_\tau^{-\gamma}e^{-\gamma\left(\mu-\frac{\sigma^2}{2}\right)(T-\tau)+\frac{\gamma^2}{2}\left((T-\tau)^2\sigma_g^2+\sigma^2(T-\tau)\right)}|\widehat{g}_\tau<\underline{g}(c),c\right]\\ &= e^{-\gamma\left(\mu-\frac{\sigma^2}{2}\right)(T-\tau)+\frac{\gamma^2}{2}\left((T-\tau)^2\sigma_g^2+\sigma^2(T-\tau)\right)}E_t\left[B_\tau^{-\gamma}|\widehat{g}_\tau<\underline{g}(c),c\right]\ .\end{aligned}$$

To compute the latter expectation, we first find the joint distribution of $b_\tau=\log(B_\tau)$ and $\widehat{g}_\tau$. Aggregate capital at τ is given by

$$\frac{B_\tau}{B_t}=e^{\mu(\tau-t)+\int_t^\tau\widehat{g}_tdt-\frac{\sigma^2}{2}(\tau-t)+\sigma\left(\widehat{Z}_\tau-\widehat{Z}_t\right)}\ .$$

Ito's Lemma implies the joint process for b_t and $\widehat{g}_t$ as

$$\begin{aligned}db_t &= \left(\mu+\widehat{g}_t-\frac{1}{2}\sigma^2\right)dt+\sigma d\widehat{Z}_t\\ d\widehat{g}_t &= \widehat{\sigma}_t^2\sigma^{-1}d\widehat{Z}_t\ .\end{aligned}$$

Integrating from t to τ:

$$\begin{aligned}b_\tau &= b_t+\mu(\tau-t)+\int_t^\tau\left(\widehat{g}_u-\frac{1}{2}\sigma^2\right)du+\sigma\int_t^\tau d\widehat{Z}_u\\ \widehat{g}_\tau &= g_t+\sigma^{-1}\int_t^\tau\widehat{\sigma}_u^2d\widehat{Z}_u\ .\end{aligned}$$

Lemma A5: The joint distribution of $(b_\tau,\widehat{g}_\tau)$ conditional on the information available at time $t<\tau$ is given by

$$\left(\begin{array}{c}b_\tau\\ \widehat{g}_\tau\end{array}\right)\sim N\left(\left(\begin{array}{c}E_t[b_\tau]\\ E_t[\widehat{g}_\tau]\end{array}\right),\left(\begin{array}{cc}V(b_\tau) & C(b_\tau,\widehat{g}_\tau)\\ C(b_\tau,\widehat{g}_\tau) & V(g_\tau)\end{array}\right)\right)$$

where

$$\begin{aligned}E_t[b_\tau] &= b_t+\left(\mu+\widehat{g}_t-\frac{1}{2}\sigma^2\right)(\tau-t)\\ E_t[\widehat{g}_\tau] &= \widehat{g}_t\\ V(b_\tau) &= (\tau-t)^2\widehat{\sigma}_t^2+\sigma^2(\tau-t)\\ V(\widehat{g}_\tau) &= \widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\\ C(b_\tau,\widehat{g}_\tau) &= \widehat{\sigma}_t^2(\tau-t)\ .\end{aligned}$$

Proof of Lemma A5: We already know that $\widehat{g}_\tau \sim N(\widehat{g}_t, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2)$. Clearly, $(b_\tau, \widehat{g}_\tau)'$ are also jointly normally distributed. We only need to find the first two moments of b_τ and its covariance with $\widehat{g}_\tau$. The mean is :

$$\begin{aligned} E_t[b_\tau] &= b_t + \mu(\tau - t) + \int_t^\tau \left(E_t[\widehat{g}_u] - \frac{1}{2}\sigma^2 \right) du \\ &= b_t + \left(\mu + \widehat{g}_t - \frac{1}{2}\sigma^2 \right)(\tau - t) \ . \end{aligned}$$

The conditional variance of b_τ as of time t is given by

$$V(b_\tau) = Var\left[\int_t^\tau \left(\mu + \widehat{g}_u - \frac{1}{2}\sigma^2 \right) du + \sigma \int_t^\tau d\widehat{Z}_u \right] \ .$$

Using Girsanov Theorem, we reexpress everything in terms of the original processes,

$$\int_t^\tau \left(\mu + \widehat{g}_u - \frac{1}{2}\sigma^2 \right) du + \sigma \int_t^\tau d\widehat{Z}_u = \int_t^\tau \left(\mu + g^{old} - \frac{1}{2}\sigma^2 \right) du + \sigma \int_t^\tau dZ_u \ ,$$

where $g^{old} \sim N(\widehat{g}_t, \widehat{\sigma}_t^2)$, given the information at time t. We then immediately obtain

$$V(b_\tau) = (\tau - t)^2 \widehat{\sigma}_t^2 + \sigma^2 (\tau - t) \ .$$

Finally, we compute the conditional covariance of b_τ with $\widehat{g}_\tau$ as of time t:

$$C(b_\tau, \widehat{g}_\tau) = E_t[b_\tau \widehat{g}_\tau] - E_t[b_\tau] E_t[\widehat{g}_\tau] \ .$$

Consider the variable

$$h_t = b_t \widehat{g}_t \ .$$

Ito's Lemma implies

$$\begin{aligned} dh_t &= db_t \widehat{g}_t + b_t d\widehat{g}_t + db_t d\widehat{g}_t \\ &= \left[\left(\mu + \widehat{g}_t - \frac{1}{2}\sigma^2 \right) \widehat{g}_t + \widehat{\sigma}_t^2 \right] dt + \left(\widehat{g}_t \sigma + b_t \widehat{\sigma}_t^2 \sigma^{-1} \right) d\widehat{Z}_t \ . \end{aligned}$$

Taking the integral on both sides:

$$h_\tau = h_t + \int_t^\tau \left[\widehat{g}_u^2 + \left(\mu - \frac{1}{2}\sigma^2 \right) \widehat{g}_u + \widehat{\sigma}_u^2 \right] du + \int_t^\tau \left(\widehat{g}_u \sigma + b_u \widehat{\sigma}_u^2 \sigma^{-1} \right) d\widehat{Z}_u \ .$$

It follows that

$$\begin{aligned} E_t[h_\tau] &= h_t + \int_t^\tau \left[E_t\left[\widehat{g}_u^2\right] + \left(\mu - \frac{1}{2}\sigma^2 \right) E_t[\widehat{g}_u] + \widehat{\sigma}_u^2 \right] du \\ &= h_t + \int_t^\tau \left[\left(\widehat{g}_t^2 + \widehat{\sigma}_t^2 - \widehat{\sigma}_u^2 \right) + \left(\mu - \frac{1}{2}\sigma^2 \right) \widehat{g}_t + \widehat{\sigma}_u^2 \right] du \\ &= h_t + \int_t^\tau \left[\widehat{g}_t^2 + \widehat{\sigma}_t^2 + \left(\mu - \frac{1}{2}\sigma^2 \right) \widehat{g}_t \right] du \\ &= h_t + \widehat{g}_t^2 (\tau - t) + \widehat{\sigma}_t^2 (\tau - t) + \left(\mu - \frac{1}{2}\sigma^2 \right) \widehat{g}_t (\tau - t) \ , \end{aligned}$$

where we used the earlier finding $E_t\left[\widehat{g}_u^2\right] = \widehat{g}_t^2 + (\widehat{\sigma}_t^2 - \widehat{\sigma}_u^2)$. The conditional covariance as of time t is given by

$$
\begin{aligned}
C\left(b_\tau, \widehat{g}_\tau\right) &= E_t\left[b_\tau \widehat{g}_\tau\right] - E_t\left[b_\tau\right] E_t\left[\widehat{g}_\tau\right] \\
&= h_t + \widehat{g}_t^2 \left(\tau - t\right) + \widehat{\sigma}_t^2 \left(\tau - t\right) + \left(\mu - \frac{1}{2}\sigma^2\right) \widehat{g}_t \left(\tau - t\right) \\
&\quad - \left(b_t + \left(\mu + \widehat{g}_t - \frac{1}{2}\sigma^2\right)\left(\tau - t\right)\right) \widehat{g}_t \\
&= \widehat{\sigma}_t^2 \left(\tau - t\right) \ .
\end{aligned}
$$

Q.E.D. (Lemma A5)

Using Lemma A5, we now finally compute $E_t\left[e^{-\gamma b_\tau} | \widehat{g}_\tau < \underline{g}(c)\right]$. Using the properties of conditional Gaussian distributions, we know that

$$
b_\tau|_{\widehat{g}_\tau = x} \sim N\left(E_t\left[b_\tau\right] + \frac{C\left(b_\tau, \widehat{g}_\tau\right)}{V\left(\widehat{g}_\tau\right)}\left(x - \widehat{g}_t\right), V\left(b_\tau\right) - \frac{C\left(b_\tau, \widehat{g}_\tau\right)^2}{V\left(\widehat{g}_\tau\right)}\right) \ .
$$

Thus

$$
\begin{aligned}
E_t\left[e^{-\gamma b_\tau} | \widehat{g}_\tau < \underline{g}(c), c\right] &= \int_{-\infty}^{\underline{g}(c)} E_t\left[e^{-\gamma b_\tau} | \widehat{g}_\tau = x\right] \phi_x\left(x | \widehat{g}_\tau < \underline{g}(c), c\right) dx \\
&= \int_{-\infty}^{\underline{g}(c)} e^{-\gamma\left(E_t[b_\tau] + \frac{C(b_\tau, \widehat{g}_\tau)}{V(\widehat{g}_\tau)}(x - \widehat{g}_t)\right) + \frac{\gamma^2}{2}\left(V(b_\tau) - \frac{C(b_\tau, \widehat{g}_\tau)^2}{V(\widehat{g}_\tau)}\right)} \phi\left(x | \widehat{g}_\tau < \underline{g}(c), c\right) dx \ ,
\end{aligned}
$$

where

$$
\phi\left(x | \widehat{g}_\tau < \underline{g}(c), c\right) = \frac{\frac{1}{\sqrt{2\pi V(\widehat{g}_\tau)}} e^{-\frac{(x - \widehat{g}_t)^2}{2V(\widehat{g}_\tau)}}}{\mathcal{N}\left(\underline{g}(c); \widehat{g}_t, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)} \ .
$$

Note that we can also write

$$
\begin{aligned}
E_t\left[e^{-\gamma b_\tau} | \widehat{g}_\tau < \underline{g}(c), c\right] &= \frac{e^{-\gamma E_t[b_\tau] + \frac{\gamma^2}{2}\left(V(b_\tau) - \frac{C(b_\tau, \widehat{g}_\tau)^2}{V(\widehat{g}_\tau)}\right)}}{\mathcal{N}\left(\underline{g}(c); \widehat{g}_t, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)} \int_{-\infty}^{\underline{g}(c)} e^{-\gamma \frac{C(b_\tau, \widehat{g}_\tau)}{V(\widehat{g}_\tau)}(x - \widehat{g}_t)} \frac{1}{\sqrt{2\pi V\left(\widehat{g}_\tau\right)}} e^{-\frac{(x - \widehat{g}_t)^2}{2V(\widehat{g}_\tau)}} dx \\
&= \frac{e^{-\gamma E_t[b_\tau] + \frac{\gamma^2}{2}\left(V(b_\tau) - \frac{C(b_\tau, \widehat{g}_\tau)^2}{V(\widehat{g}_\tau)}\right)}}{\mathcal{N}\left(\underline{g}(c); \widehat{g}_t, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)} \int_{-\infty}^{\underline{g}(c)} \frac{1}{\sqrt{2\pi V\left(\widehat{g}_\tau\right)}} e^{\frac{-(x - \widehat{g}_t)^2 - 2\gamma C(b_\tau, \widehat{g}_\tau)(x - \widehat{g}_t)}{2V(\widehat{g}_\tau)}} dx \\
&= \frac{e^{-\gamma E_t[b_\tau] + \frac{\gamma^2}{2}\left(V(b_\tau) - \frac{C(b_\tau, \widehat{g}_\tau)^2}{V(\widehat{g}_\tau)}\right)}}{\mathcal{N}\left(\underline{g}(c); \widehat{g}_t, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)} e^{\frac{\gamma^2 C(b_\tau, \widehat{g}_\tau)^2}{2V(\widehat{g}_\tau)}} \int_{-\infty}^{\underline{g}(c)} \frac{1}{\sqrt{2\pi V\left(\widehat{g}_\tau\right)}} e^{\frac{-(x - (\widehat{g}_t - \gamma C(b_\tau, \widehat{g}_\tau)))^2}{2V(\widehat{g}_\tau)}} dx \\
&= \frac{\mathcal{N}\left(\underline{g}(c); \widehat{g}_t - \gamma C(b_\tau, \widehat{g}_\tau), \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)}{\mathcal{N}\left(\underline{g}(c); \widehat{g}_t, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)} \times e^{-\gamma E_t[b_\tau] + \frac{\gamma^2}{2} V(b_\tau)} \ .
\end{aligned}
$$

In conclusion, substituting for $E_t[b_\tau]$, $V[\widehat{g}_\tau]$ and $C(b_\tau,\widehat{g}_\tau)$, we obtain

$$E_t\left[e^{-\gamma b_\tau}|\widehat{g}_\tau<\underline{g}(c),c\right] = B_t^{-\gamma}e^{-\gamma\left(\mu+\widehat{g}_t-\frac{1}{2}\sigma^2\right)(\tau-t)+\frac{\gamma^2}{2}\left((\tau-t)^2\widehat{\sigma}_t^2+\sigma^2(\tau-t)\right)}\frac{\mathcal{N}\left(\underline{g}(c),\widehat{g}_t-\gamma\widehat{\sigma}_t^2\left(\tau-t\right),\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)}{\mathcal{N}\left(\underline{g}(c);\widehat{g}_t,\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)}\ .$$

Finally, putting all terms together,

$$\begin{aligned}E_t\left[\pi_{\tau+}|\widehat{g}_\tau<\underline{g}(c),c\right] &= B_t^{-\gamma}e^{-\gamma\mu(T-t)-\gamma\widehat{g}_t(\tau-t)+\gamma\frac{\sigma^2}{2}(T-t)+\frac{\gamma^2}{2}\left((T-\tau)^2\widehat{\sigma}_g^2+(\tau-t)^2\widehat{\sigma}_t^2\right)+\frac{\gamma^2}{2}\sigma^2(T-t)}\times\\ &\quad\times\frac{\mathcal{N}\left(\underline{g}(c),\widehat{g}_t-\gamma\widehat{\sigma}_t^2\left(\tau-t\right),\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)}{\mathcal{N}\left(\underline{g}(c);\widehat{g}_t,\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)}\ ,\end{aligned}$$

concluding Case 1.

CASE 2: NO POLICY CHANGE: The state price density right after time τ if there is no policy change is given in (B24) as follows (again dropping λ^{-1} and using $B_\tau=B_{\tau+}$):

$$\pi_{\tau+}=B_\tau^{-\gamma}e^{-\gamma\left(\mu+\widehat{g}_\tau-\frac{\sigma^2}{2}\right)(T-\tau)+\frac{\gamma^2}{2}\left((T-\tau)^2\widehat{\sigma}_\tau^2+\sigma^2(T-\tau)\right)}\ .$$

The main difference from Case 1 is that $\widehat{g}_\tau$ now also enters the exponent, as a result of which we cannot factor it out. Consider the conditional expectation

$$\begin{aligned}E_t\left[\pi_{\tau+}|\widehat{g}_\tau>\underline{g}(c),c\right] &= E_t\left[B_\tau^{-\gamma}e^{-\gamma\left(\mu+\widehat{g}_\tau-\frac{\sigma^2}{2}\right)(T-\tau)+\frac{\gamma^2}{2}\left((T-\tau)^2\widehat{\sigma}_\tau^2+\sigma^2(T-\tau)\right)}|\widehat{g}_\tau>\underline{g}(c),c\right]\\ &= e^{-\gamma\mu(T-\tau)+\gamma\frac{\sigma^2}{2}(T-\tau)+\frac{\gamma^2}{2}\left((T-\tau)^2\widehat{\sigma}_\tau^2+\sigma^2(T-\tau)\right)}E_t\left[e^{-\gamma(\widehat{g}_\tau(T-\tau)+b_\tau)}|\widehat{g}_\tau>\underline{g}(c),c\right]\end{aligned}$$

Define

$$y_\tau\equiv\widehat{g}_\tau\left(T-\tau\right)+b_\tau\ .$$

We can now use Lemma A5, designed for the joint density of $(b_\tau,\widehat{g}_\tau)$, to compute the joint density of $(y_\tau,\widehat{g}_\tau)$. We obtain

$$\left(\begin{array}{c}y_\tau\\ \widehat{g}_\tau\end{array}\right)\sim N\left(\left(\begin{array}{c}\widehat{g}_\tau\left(T-\tau\right)+E_t\left[b_\tau\right]\\ E_t\left[\widehat{g}_\tau\right]\end{array}\right)\left(\begin{array}{cc}V\left(y_\tau\right) & C\left(y_\tau,\widehat{g}_\tau\right)\\ C\left(y_\tau,\widehat{g}_\tau\right) & V\left(\widehat{g}_\tau\right)\end{array}\right)\right)\ .$$

The only two new terms to compute are $V(y_\tau)$ and $C(y_\tau,\widehat{g}_\tau)$. We have

$$\begin{aligned}V\left(y_\tau\right) &= \left(T-\tau\right)^2V\left(\widehat{g}_\tau\right)+V\left(b_\tau\right)+2\left(T-\tau\right)C\left(b_\tau,\widehat{g}_\tau\right)\\ &= \left(T-\tau\right)^2\left(\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)+\left(\tau-t\right)^2\widehat{\sigma}_t^2+\sigma^2\left(\tau-t\right)+2\left(T-\tau\right)\widehat{\sigma}_t^2\left(\tau-t\right)\\ &= \left(T-\tau\right)^2\widehat{\sigma}_t^2-\left(T-\tau\right)^2\widehat{\sigma}_\tau^2+\left(\tau-t\right)^2\widehat{\sigma}_t^2+\sigma^2\left(\tau-t\right)+2\left(T-\tau\right)\widehat{\sigma}_t^2\left(\tau-t\right)\\ &= \widehat{\sigma}_t^2\left[\left(T-\tau\right)^2+\left(\tau-t\right)^2+2\left(T-\tau\right)\left(\tau-t\right)\right]-\left(T-\tau\right)^2\widehat{\sigma}_\tau^2+\sigma^2\left(\tau-t\right)\\ &= \widehat{\sigma}_t^2\left(T-t\right)^2-\left(T-\tau\right)^2\widehat{\sigma}_\tau^2+\sigma^2\left(\tau-t\right)\end{aligned}$$

and

$$\begin{aligned}C\left(y_\tau,\widehat{g}_\tau\right) &= \left(T-\tau\right)V\left(\widehat{g}_\tau\right)+C\left(b_\tau,\widehat{g}_\tau\right)=\left(T-\tau\right)\left(\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)+\widehat{\sigma}_t^2\left(\tau-t\right)\\ &= \left[\widehat{\sigma}_t^2\left(T-t\right)-\left(T-\tau\right)\widehat{\sigma}_\tau^2\right]\ .\end{aligned}$$

It follows from the properties of conditional Gaussian distributions that

$$y_\tau|_{\widehat{g}_\tau = x} \sim N\left(E_t\left[y_\tau\right] + \frac{C\left(y_\tau, \widehat{g}_\tau\right)}{V\left(\widehat{g}_\tau\right)}\left(x - \widehat{g}_t\right), V\left(y_\tau\right) - \frac{C\left(y_\tau, \widehat{g}_\tau\right)^2}{V\left(\widehat{g}_\tau\right)}\right) .$$

Thus

$$\begin{aligned} E_t\left[e^{-\gamma y_\tau}|\widehat{g}_\tau > \underline{g}(c), c\right] &= \int_{\underline{g}(c)}^{\infty} E\left[e^{-\gamma y_\tau}|\widehat{g}_\tau = x\right] \phi\left(x|\widehat{g}_\tau > \underline{g}(c), c\right) dx \\ &= \int_{\underline{g}(c)}^{\infty} e^{-\gamma\left(E_t[y_\tau] + \frac{C(y_\tau, \widehat{g}_\tau)}{V(\widehat{g}_\tau)}(x - \widehat{g}_t)\right) + \frac{\gamma^2}{2}\left(V(y_\tau) - \frac{C(y_\tau, \widehat{g}_\tau)^2}{V(\widehat{g}_\tau)}\right)} \phi\left(x|\widehat{g}_\tau > \underline{g}(c), c\right) dx \ , \end{aligned}$$

where

$$\phi\left(x|\widehat{g}_\tau > \underline{g}(c), c\right) = \frac{\frac{1}{\sqrt{2\pi V(\widehat{g}_\tau)}} e^{-\frac{(x - \widehat{g}_t)^2}{2V(\widehat{g}_\tau)}}}{1 - \mathcal{N}\left(\underline{g}(c); \widehat{g}_t, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)} .$$

We therefore obtain

$$E_t\left[e^{-\gamma y_\tau}|\widehat{g}_\tau > \underline{g}(c), c\right] = \frac{e^{-\gamma E_t[y_\tau] + \frac{\gamma^2}{2}\left(V(y_\tau) - \frac{C(y_\tau, \widehat{g}_\tau)^2}{V(\widehat{g}_\tau)}\right)}}{1 - \mathcal{N}\left(\underline{g}(c); \widehat{g}_t, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)} \int_{\underline{g}(c)}^{\infty} e^{-\gamma \frac{C(y_\tau, \widehat{g}_\tau)}{V(\widehat{g}_\tau)}(x - \widehat{g}_t)} \frac{1}{\sqrt{2\pi V\left(\widehat{g}_\tau\right)}} e^{-\frac{(x - \widehat{g}_t)^2}{2V(\widehat{g}_\tau)}} dx .$$

The same steps as in Case 1 yield

$$E_t\left[e^{-\gamma y_\tau}|\widehat{g}_\tau > \underline{g}(c), c\right] = e^{-\gamma(\widehat{g}_t(T-\tau) + E_t[b_\tau]) + \frac{\gamma^2}{2}V(y_\tau)} \frac{\left(1 - \mathcal{N}\left(\underline{g}(c), \widehat{g}_t - \gamma C\left(y_\tau, \widehat{g}_\tau\right), \widehat{\sigma}_t^2 - \sigma_\tau^2\right)\right)}{1 - \mathcal{N}\left(\underline{g}(c), \widehat{g}_t, \widehat{\sigma}_t^2 - \sigma_\tau^2\right)} .$$

Finally, putting all terms together,

$$\begin{aligned} E_t\left[\pi_{\tau+}|\widehat{g}_\tau > \underline{g}(c), c\right] &= e^{-\gamma\mu(T-\tau) + \gamma\frac{\sigma^2}{2}(T-\tau) + \frac{\gamma^2}{2}\left((T-\tau)^2\widehat{\sigma}_\tau^2 + \sigma^2(T-\tau)\right)} E_t\left[e^{-\gamma y_\tau}|\widehat{g}_\tau > \underline{g}(c), c\right] \\ &= B_t^{-\gamma} e^{-\gamma\mu(T-t) + \frac{\gamma^2}{2}\left((T-t)^2\widehat{\sigma}_t^2 + \sigma^2(T-\tau)\right)} e^{-\gamma\left(\widehat{g}_t(T-t) - \frac{1}{2}\sigma^2(T-t)\right)} \times \\ &\quad \times \frac{\left(1 - \mathcal{N}\left(\underline{g}(c), \widehat{g}_t - \gamma C\left(y_\tau, \widehat{g}_\tau\right), \widehat{\sigma}_t^2 - \sigma_\tau^2\right)\right)}{1 - \mathcal{N}\left(\underline{g}(c), \widehat{g}_t, \widehat{\sigma}_t^2 - \sigma_\tau^2\right)} . \end{aligned}$$

STEP 2: We now integrate out the cost c. The conditional expectation is

$$\begin{aligned} E_t\left[\pi_{\tau+}|c\right] &= p_t\left(c\right) E_t\left[\pi_{\tau+}|\widehat{g}_\tau < \underline{g}\left(c\right), c\right] + \left(1 - p_t\left(c\right)\right) E_t\left[\pi_{\tau+}|\widehat{g}_\tau > \underline{g}\left(c\right), c\right] \\ &= B_t^{-\gamma}\mathcal{N}\left(\underline{g}\left(c\right), \widehat{g}_t - \gamma\widehat{\sigma}_t^2\left(\tau - t\right), \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right) \times \\ &\quad \times e^{-\gamma\mu(T-t) - \gamma\widehat{g}_t(\tau-t) + \gamma\frac{\sigma^2}{2}(T-t) + \frac{\gamma^2}{2}\left((T-\tau)^2\sigma_g^2 + (\tau-t)^2\widehat{\sigma}_t^2\right) + \frac{\gamma^2}{2}\sigma^2(T-t)} \\ &\quad + B_t^{-\gamma}\left(1 - \mathcal{N}\left(\underline{g}\left(c\right), \widehat{g}_t - \gamma C\left(y_\tau, \widehat{g}_\tau\right), \widehat{\sigma}_t^2 - \sigma_\tau^2\right)\right) \times \\ &\quad \times e^{-\gamma\mu(T-t) + \frac{\gamma^2}{2}\left((T-t)^2\widehat{\sigma}_\tau^2 + \sigma^2(T-\tau)\right)} e^{-\gamma\left(\widehat{g}_t(T-t) - \frac{1}{2}\sigma^2(T-t)\right)} \end{aligned}$$

Note that the cost c only enters into the two terms $\mathcal{N}\left(\underline{g}(c), \widehat{g}_t - \gamma\widehat{\sigma}_t^2(\tau - t), \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)$ and $\mathcal{N}\left(\underline{g}(c), \widehat{g}_t - \gamma C(y_\tau, \widehat{g}_\tau), \widehat{\sigma}_t^2 - \sigma_\tau^2\right)$. Since the distribution of c is independent of everything else, we obtain

$$\begin{aligned} E_t[\pi_{\tau+}] &= B_t^{-\gamma} E_t\left[\mathcal{N}\left(\underline{g}(c), \widehat{g}_t - \gamma\widehat{\sigma}_t^2(\tau - t), \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)\right] \times \\ &\quad \times e^{-\gamma\mu(T-t) - \gamma\widehat{g}_t(\tau - t) + \gamma\frac{\sigma^2}{2}(T-t) + \frac{\gamma^2}{2}\left((T-\tau)^2\sigma_g^2 + (\tau - t)^2\widehat{\sigma}_t^2\right) + \frac{\gamma^2}{2}\sigma^2(T-t)} \\ &\quad + B_t^{-\gamma}\left(1 - E_t\left[\mathcal{N}\left(\underline{g}(c), \widehat{g}_t - \gamma C(y_\tau, \widehat{g}_\tau), \widehat{\sigma}_t^2 - \sigma_\tau^2\right)\right]\right) \times \\ &\quad \times e^{-\gamma\mu(T-t) + \frac{\gamma^2}{2}\left((T-t)^2\widehat{\sigma}_\tau^2 + \sigma^2(T-\tau)\right)} e^{-\gamma\left(\widehat{g}_t(T-t) - \frac{1}{2}\sigma^2(T-t)\right)} . \end{aligned}$$

Finally, using the law of iterated expectations, we can compute

$$\begin{aligned} p_t^{yes} &\equiv E_t\left[\mathcal{N}\left(\underline{g}(c), \widehat{g}_t - \gamma\widehat{\sigma}_t^2(\tau - t), \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)\right] = E_t\left[\Pr\left(x < \underline{g}(c)\,|c\right)\right] \\ &= E_t\left[E_t[1_{x<\underline{g}(c)}|c]\right] = E_t\left[1_{x<\underline{g}(c)}\right] = \Pr\left(x < \underline{g}(c)\right) \\ &= \Pr\left(x < \underline{g}(0) - \frac{c}{(\gamma - 1)(T - \tau)}\right) \\ &= \mathcal{N}\left(\underline{g}(0); \widehat{g}_t - \gamma\widehat{\sigma}_t^2(\tau - t) - \frac{\frac{1}{2}\sigma_c^2}{(\gamma - 1)(T - \tau)}; \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2 + \frac{\sigma_c^2}{(\gamma - 1)^2(T - \tau)^2}\right) . \end{aligned}$$

Similarly,

$$\begin{aligned} p_t^{no} &\equiv 1 - E_t\left[\mathcal{N}\left(\underline{g}(c), \widehat{g}_t - \gamma C(y_\tau, b_\tau), \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2\right)\right] \\ &= 1 - \mathcal{N}\left(\underline{g}(0); \widehat{g}_t - \gamma C(y_\tau, b_\tau) - \frac{\frac{1}{2}\sigma_c^2}{(\gamma - 1)(T - \tau)}; \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2 + \frac{\sigma_c^2}{(\gamma - 1)^2(T - \tau)^2}\right) . \end{aligned}$$

Substituting in $E_t[\pi_{\tau+}]$, the claim of Proposition A1 follows.

QED (Proposition A1)

Proposition 6. *The stochastic discount factor (SDF) follows the process*

$$\frac{d\pi_t}{\pi_t} = -\sigma_{\pi,t} d\widehat{Z}_t + J_\pi 1_{\{t=\tau\}} , \tag{B36}$$

where $d\widehat{Z}_t$ is the Brownian motion from Proposition 1, $1_{\{t=\tau\}}$ is an indicator function equal to one for $t = \tau$ and zero otherwise, and the jump component J_π is given by

$$J_\pi = \begin{cases} J_\pi^{yes} = \frac{(1-p_\tau)(1-F(\widehat{g}_\tau))}{p_\tau + (1-p_\tau)F(\widehat{g}_\tau)} & \text{if policy changes} \\ J_\pi^{no} = \frac{p_\tau(F(\widehat{g}_\tau)-1)}{p_\tau + (1-p_\tau)F(\widehat{g}_\tau)} & \text{if policy does not change .} \end{cases} \tag{B37}$$

For $t > \tau$, $\sigma_{\pi,t}$ is given by

$$\sigma_{\pi,t} = \gamma\left[\sigma + (T - t)\,\widehat{\sigma}_t^2\sigma^{-1}\right] , \tag{B38}$$

and for $t \leq \tau$, it is given by (Corollary A1 in the paper):

$$\sigma_{\pi,t} = \gamma\sigma - \frac{1}{\Omega(\widehat{g}_t, t)}\frac{\partial\Omega(\widehat{g}_t, t)}{\partial\widehat{g}_t}\widehat{\sigma}_t^2\sigma^{-1} . \tag{B39}$$

Proof of Proposition 6: For $t < \tau$, the SDF dynamics stem from an application of Ito's Lemma to (B35), which also yields the volatility (B39). Because the state price density is a martingale, $\pi_t = E_t[\pi_T]$, the drift of the process is zero. For $t > \tau$, the state price density is given in closed form as in equation (B24), with t in place of τ in the formula. An application of Ito's Lemma immediately leads to the diffusion term (B38).

At the time of the anouncement, τ, the state price density jumps from (B25) to either of the two expressions in (B24). We obtain the size of the jump by computing the difference:

$$\begin{aligned} J_{\pi,\tau}^{yes} &= \left(\frac{\pi_{\tau+}^{yes}}{\pi_\tau} - 1\right) = \frac{\pi_{\tau+}^{yes}}{p_\tau \pi_{\tau+}^{yes} + (1-p_\tau)\pi_{\tau+}^{no}} - 1 = \frac{1}{p_\tau + (1-p_\tau)\frac{\pi_{\tau+}^{no}}{\pi_{\tau+}^{yes}}} - 1 \\ &= \frac{1}{p_\tau + (1-p_\tau) e^{-\gamma\widehat{g}(T-\tau)-\frac{1}{2}\gamma^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}} - 1 = \frac{(1-p_\tau)\left(1 - e^{-\gamma\widehat{g}_\tau(T-\tau)-\frac{1}{2}\gamma^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}\right)}{p_\tau + (1-p_\tau) e^{-\gamma\widehat{g}(T-\tau)-\frac{1}{2}\gamma^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}} . \end{aligned}$$

The expression for the jump if there is no policy change follows from the martingale condition

$$E_\tau[J_\pi] = p_\tau J_{\pi,\tau}^{yes} + (1-p_\tau) J_{\pi,\tau}^{no} = 0 ,$$

which implies

$$J_{\pi,\tau}^{no} = -\frac{p_\tau}{(1-p_\tau)} J_{\pi,\tau}^{yes} = \frac{p_\tau\left(e^{-\gamma\widehat{g}_\tau(T-\tau)-\frac{1}{2}\gamma^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)} - 1\right)}{p_\tau + (1-p_\tau) e^{-\gamma\widehat{g}(T-\tau)-\frac{1}{2}\gamma^2(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}} .$$

QED (Proposition 6)

Proposition A2. In the benchmark model for $t \leq \tau$, the stock price for firm i is given by

$$M_t^i = B_t^i \frac{\Phi(\widehat{g}_t, t)}{\Omega(\widehat{g}_t, t)} , \qquad \text{(B40)}$$

where

$$\begin{aligned} \Phi(\widehat{g}_t, t) &= \overline{p}_t^{yes} K_t^{yes} + (1-\overline{p}_t^{no}) K_t^{no} \\ K_t^{yes} &= e^{(1-\gamma)\mu(T-t)+(1-\gamma)\widehat{g}_t(\tau-t)+\frac{(1-\gamma)^2}{2}\left((T-\tau)^2\sigma_g^2+(\tau-t)^2\widehat{\sigma}_t^2\right)-(1-\gamma)\gamma\frac{\sigma^2}{2}(T-t)} \\ K_t^{no} &= e^{(1-\gamma)\mu(T-t)+(1-\gamma)\widehat{g}_t(T-t)+\frac{(1-\gamma)^2}{2}\widehat{\sigma}_t^2(T-t)^2-(1-\gamma)\gamma\frac{\sigma^2}{2}(T-t)} \end{aligned}$$

and

$$\begin{aligned} \overline{p}_t^{yes} &= N\left(\underline{g}(0); \widehat{g}_t + (1-\gamma)\widehat{\sigma}_t^2(\tau-t) + \frac{\sigma_c^2/2}{(T-\tau)(1-\gamma)}, \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2 + \frac{\sigma_c^2}{(T-\tau)^2(1-\gamma)^2}\right) \\ \overline{p}_t^{no} &= N\left(\underline{g}(0); \widehat{g}_t + (1-\gamma)\left[\widehat{\sigma}_t^2(T-t) - (T-\tau)\widehat{\sigma}_\tau^2\right] + \frac{\sigma_c^2/2}{(T-\tau)(1-\gamma)}, \widehat{\sigma}_t^2 - \sigma_\tau^2 + \frac{\sigma_c^2}{(T-\tau)^2(1-\gamma)^2}\right) \end{aligned}$$

Proof of Proposition A2: The proof is similar to that of Proposition A1. For $t < \tau$, market value satisfies $M_t^i = \frac{E_t\left[\pi_{\tau+} M_{\tau+}^i\right]}{\pi_t}$. We need to compute the numerator. As in Proposition A1, we proceed in two steps. In the first step, we compute the conditional expectations, and in the second step, we integrate out c:

$$E_t\left[\pi_{\tau+}M^i_{\tau+}|c\right] = p_t(c)\,E_t\left[\pi_{\tau+}M^i_{\tau+}|\widehat{g}_\tau < \underline{g}(c),c\right] + (1-p_t(c))\,E_t\left[\pi_{\tau+}M^i_{\tau+}|\widehat{g}_\tau > \underline{g}(c),c\right] \;.$$

CASE 1. POLICY CHANGE. Using the expressions in (B24) and (B20):

$$\begin{aligned} E_t\left[\pi_{\tau+}M^i_{\tau+}|\widehat{g}_\tau < \underline{g}(c),c\right] &= E_t\Bigg[B_\tau^{-\gamma}e^{-\gamma\left(\mu-\frac{\sigma^2}{2}\right)(T-\tau)+\frac{\gamma^2}{2}\left((T-\tau)^2\sigma_g^2+\sigma^2(T-\tau)\right)} \\ &\quad \times B^i_\tau e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-\tau)+\frac{(1-2\gamma)}{2}\left((T-\tau)^2\sigma_g^2+\sigma^2(T-t)\right)}|\widehat{g}_\tau < \underline{g}(c),c\Bigg] \\ &= e^{(1-\gamma)\left(\mu-\frac{\sigma^2}{2}\right)(T-\tau)+\frac{(1-\gamma)^2}{2}\left((T-\tau)^2\sigma_g^2+\sigma^2(T-\tau)\right)}E_t\left[B_\tau^{-\gamma}B^i_\tau|\widehat{g}_\tau < \underline{g}(c),c\right] \;. \end{aligned}$$

Under the original probability measure, $B_\tau = B_t e^{\mu(\tau-t)+g^{old}(\tau-t)-\frac{1}{2}\sigma^2(\tau-t)+\sigma(Z_\tau-Z_t)}$, while

$$\begin{aligned} B^i_\tau &= B^i_t e^{\mu(\tau-t)+g^{old}(\tau-t)-\frac{1}{2}\sigma^2(\tau-t)+\sigma(Z_\tau-Z_t)-\frac{1}{2}\sigma_1^2(\tau-t)+\sigma_1\left(Z^i_\tau-Z^i_t\right)} \\ &= B^i_t\left(\frac{B_\tau}{B_t}\right)e^{-\frac{1}{2}\sigma_1^2(\tau-t)+\sigma_1\left(Z^i_\tau-Z^i_t\right)} \;. \end{aligned} \qquad \text{(B41)}$$

Thus

$$\begin{aligned} E_t\left[B_\tau^{-\gamma}B^i_\tau|\widehat{g}_\tau < \underline{g}(c),c\right] &= \frac{B^i_t}{B_t}E_t\left[B_\tau^{1-\gamma}e^{-\frac{1}{2}\sigma_1^2(\tau-t)+\sigma_1\left(Z^i_\tau-Z^i_t\right)}|\widehat{g}_\tau < \underline{g}(c),c\right] \\ &= \frac{B^i_t}{B_t}E_t\left[B_\tau^{1-\gamma}|\widehat{g}_\tau < \underline{g}(c),c\right] \end{aligned}$$

since the Brownian motions Z^i are independent of B_τ. The closed-form expression for the expectation is then identical to that obtained in Proposition A1 when we substitute $(1-\gamma)$ for $-\gamma$. That is:

$$\begin{aligned} E_t\left[B_\tau^{1-\gamma}|\widehat{g}_\tau < \underline{g}(c),c\right] &= B_t^{1-\gamma}e^{(1-\gamma)\left(\mu+\widehat{g}_t-\frac{1}{2}\sigma^2\right)(\tau-t)+\frac{(1-\gamma)^2}{2}\left((\tau-t)^2\widehat{\sigma}_t^2+\sigma^2(\tau-t)\right)}\times \\ &\quad \times\frac{\mathcal{N}\left(\underline{g}(c),\widehat{g}_t+(1-\gamma)\,\widehat{\sigma}_t^2\,(\tau-t)\,,\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)}{\mathcal{N}\left(\underline{g}(c);\widehat{g}_t,\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)} \;. \end{aligned}$$

Substitute to obtain

$$\begin{aligned} E_t\left[\pi_{\tau+}M^i_{\tau+}|\widehat{g}_\tau < \underline{g}(c),c\right] &= B^i_tB_t^{-\gamma}e^{(1-\gamma)\mu(T-t)+(1-\gamma)\widehat{g}_t(\tau-t)-(1-\gamma)\gamma\frac{\sigma^2}{2}(T-t)+\frac{(1-\gamma)^2}{2}\left((T-\tau)^2\sigma_g^2+(\tau-t)^2\widehat{\sigma}_t^2\right)}\times \\ &\quad \times\frac{\mathcal{N}\left(\underline{g}(c),\widehat{g}_t+(1-\gamma)\,\widehat{\sigma}_t^2\,(\tau-t)\,,\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)}{\mathcal{N}\left(\underline{g}(c);\widehat{g}_t,\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)} \;. \end{aligned}$$

CASE 2. NO POLICY CHANGE

$$\begin{aligned} E_t\left[\pi_{\tau+}M^i_{\tau+}|\widehat{g}_\tau > \underline{g}(c),c\right] &= E_t\Bigg[B_\tau^{-\gamma}e^{-\gamma\left(\mu+\widehat{g}_\tau-\frac{\sigma^2}{2}\right)(T-\tau)+\frac{\gamma^2}{2}\left((T-\tau)^2\widehat{\sigma}_\tau^2+\sigma^2(T-\tau)\right)} \\ &\quad \times B^i_\tau e^{\left(\mu+\widehat{g}_\tau-\frac{\sigma^2}{2}\right)(T-\tau)+\frac{(1-2\gamma)}{2}\left((T-\tau)^2\widehat{\sigma}_\tau^2+\sigma^2(T-\tau)\right)}|\widehat{g}_\tau > \underline{g}(c),c\Bigg] \\ &= e^{(1-\gamma)\mu(T-\tau)-(1-\gamma)\frac{\sigma^2}{2}(T-\tau)+\frac{(1-\gamma)^2}{2}\sigma^2(T-\tau)+\frac{(1-\gamma)^2}{2}(T-\tau)^2\widehat{\sigma}_\tau^2} \\ &\quad \times E_t\left[B_\tau^{-\gamma}B^i_\tau e^{(1-\gamma)\widehat{g}_\tau(T-\tau)}|\widehat{g}_\tau > \underline{g}(c),c\right] \;. \end{aligned}$$

Using (B41) again, we have

$$\begin{aligned} E_t\left[B_\tau^{-\gamma}B_\tau^i e^{(1-\gamma)\widehat{g}_\tau(T-\tau)}|\widehat{g}_\tau > \underline{g}(c), c\right] &= \frac{B_t^i}{B_t}E_t\left[B_\tau^{1-\gamma}e^{(1-\gamma)\widehat{g}_\tau(T-\tau)}|\widehat{g}_\tau > \underline{g}(c), c\right] \\ &= \frac{B_t^i}{B_t}E_t\left[e^{(1-\gamma)(b_\tau+\widehat{g}_\tau(T-\tau))}|\widehat{g}_\tau > \underline{g}(c), c\right] \end{aligned}$$

where the idiosyncratic Brownian motion term $(Z_\tau^i - Z_t^i)$ is independent of both B_τ and $\widehat{g}_\tau$. Recalling the notation $y_\tau = b_\tau + \widehat{g}_\tau(T-\tau)$, we can use the results in Proposition A1 with $-\gamma$ replaced by $(1-\gamma)$ to obtain

$$\begin{aligned} E_t\left[e^{(1-\gamma)(b_\tau+\widehat{g}_\tau(T-\tau))}|\widehat{g}_\tau > \underline{g}(c), c\right] &= e^{(1-\gamma)(\widehat{g}_t(T-\tau)+E_t[b_\tau])+\frac{(1-\gamma)^2}{2}V(y_\tau)} \times \\ &\times \frac{\left(1-\mathcal{N}\left(\underline{g}(c), \widehat{g}_t + (1-\gamma)\,C\left(y_\tau, \widehat{g}_\tau\right), \widehat{\sigma}_t^2 - \sigma_\tau^2\right)\right)}{1-\mathcal{N}\left(\underline{g}(c), \widehat{g}_t, \widehat{\sigma}_t^2 - \sigma_\tau^2\right)} . \end{aligned}$$

Putting the terms together,

$$\begin{aligned} E_t\left[\pi_{\tau+}M_{\tau+}^i|\widehat{g}_\tau > \underline{g}(c), c\right] &= B_t^i B_t^{-\gamma} e^{(1-\gamma)\mu(T-t)-\gamma\frac{(1-\gamma)}{2}\sigma^2(T-t)+(1-\gamma)\widehat{g}_t(T-t)+\frac{(1-\gamma)^2}{2}\widehat{\sigma}_t^2(T-t)^2} \times \\ &\times \frac{\left(1-\mathcal{N}\left(\underline{g}(c), \widehat{g}_t + (1-\gamma)\,C\left(y_\tau, \widehat{g}_\tau\right), \widehat{\sigma}_t^2 - \sigma_\tau^2\right)\right)}{1-\mathcal{N}\left(\underline{g}(c), \widehat{g}_t, \widehat{\sigma}_t^2 - \sigma_\tau^2\right)} . \end{aligned}$$

The last step is to integrate out the random cost c. Using the same steps as in the proof of Proposition A1, we obtain $\overline{p}^{yes}$ and $\overline{p}^{no}$ as in the claim of the proposition.

QED (Proposition A2)

Proposition 7. *The return process for stock i is given by*

$$\frac{dM_t^i}{M_t^i} = \mu_{M,t}dt + \sigma_{M,t}d\widehat{Z}_t + \sigma_1 dZ_t^i + J_M 1_{\{t=\tau\}} , \tag{B42}$$

where the jump component J_M is given by

$$J_M = \begin{cases} J_M^{yes} = R\left(\widehat{g}_\tau\right) & \text{if policy changes} \\ J_M^{no} = R\left(\widehat{g}_\tau\right)G\left(\widehat{g}_\tau\right) + G\left(\widehat{g}_\tau\right) - 1 & \text{if policy does not change .} \end{cases} \tag{B43}$$

For $t > \tau$, we have

$$\mu_{M,t} = \gamma\left[\sigma + (T-t)\,\widehat{\sigma}_t^2\sigma^{-1}\right]^2 \tag{B44}$$

$$\sigma_{M,t} = \sigma + (T-t)\,\widehat{\sigma}_t^2\sigma^{-1} , \tag{B45}$$

and for $t \leq \tau$, $\mu_{M,t}$ and $\sigma_{M,t}$ are given by (Corollary A2 in the paper)

$$\sigma_{M,t} = \sigma + \left(\frac{\partial\Phi\left(\widehat{g}_t, t\right)/\partial\widehat{g}_t}{\Phi\left(\widehat{g}_t, t\right)} - \frac{\partial F\left(\widehat{g}_t, t\right)/\partial\widehat{g}_t}{F\left(\widehat{g}_t, t\right)}\right)\widehat{\sigma}_t^2\sigma^{-1} \tag{B46}$$

$$\mu_M = \sigma_{\pi,t}\sigma_{M,t} , \tag{B47}$$

where $\sigma_{\pi,t}$ and $\sigma_{M,t}$ are given in equations (B39) and (B46), respectively.

Proof of Proposition 7: For $t < \tau$, the dynamics of the return dM_t^i/M_t^i stem from an application of Ito's Lemma to (B40), which also yields the volatility (B46). The expected return is given by $\mu_M = -Cov(d\pi_t/\pi_t, dM_t^i/M_t^i) = \sigma_{\pi,t}\sigma_{M,t}$, which yields (B47).

For $t > \tau$, the price is given in closed form as in equation (B20), with t in place of τ in the formula. An application of Ito's Lemma immediately yields the diffusion term (B45). We obtain (B44) by using the closed-form solution for the SDF diffusion in (B38).

At time τ, stock prices jump. The jump size in case of a policy change, $J_M^{yes} = R(\widehat{g}_\tau)$, is already derived in Proposition 3. For J_M^{no}, we have

$$J_M^{no} = \frac{M_{\tau+}^{i,no} - M_\tau^i}{M_\tau^i} = \frac{-\omega(M_{\tau+}^{i,yes} - M_{\tau+}^{i,no})}{\omega M_{\tau+}^{i,yes} + (1-\omega)M_{\tau+}^{i,no}} = \frac{-\omega(1 - M_{\tau+}^{i,no}/M_{\tau+}^{i,yes})}{\omega + (1-\omega)(M_{\tau+}^{i,no}/M_{\tau+}^{i,yes})} \quad \text{(B48)}$$

$$= \frac{p_\tau(G(\widehat{g}_\tau) - 1)}{p_\tau + (1-p_\tau)F(\widehat{g}_\tau)G(\widehat{g}_\tau)}, \quad \text{(B49)}$$

where we use ω in (B22) and $G(\widehat{g}_\tau) = M_{\tau+}^{i,no}/M_{\tau+}^{i,yes}$. This is the same expression as in (B43), as can be easily verified by substituting for $R(\widehat{g}_\tau)$ from (B26).

QED (Proposition 7)

Corollary 3. *The market value of each firm increases at the announcement of no policy change (i.e., $J_M^{no} > 0$) if and only if $\widehat{g}_\tau > g^*$, where g^* is given in equation (B28).*

Proof of Corollary 3: The proof follows immediately from expression (B49), which is positive if and only if $G(\widehat{g}_\tau) > 1$, where $G(\widehat{g}_\tau)$ is given in (B18).

QED (Corollary 3)

Proposition 8. *The conditional expected jump in stock prices at time τ, as perceived by investors just before time τ, is given by*

$$\mathrm{E}_\tau(J_M) = -\frac{p_\tau(1-p_\tau)(1-F(\widehat{g}_\tau))(1-G(\widehat{g}_\tau))}{p_\tau + (1-p_\tau)F(\widehat{g}_\tau)G(\widehat{g}_\tau)}. \quad \text{(B50)}$$

Proof of Proposition 8: The proof follows immediately from $\mathrm{E}_\tau(J_M) = p_\tau J_M^{yes} + (1-p_\tau)J_M^{no}$ and substituting $J_M^{yes} = R(\widehat{g}_\tau)$ in (B16) and J_M^{no} in (B49).

QED (Proposition 8)

Corollary 4. *We have $\mathrm{E}_\tau(J_M) < 0$ if and only if*

$$g^* < \widehat{g}_\tau < g^{**}, \quad \text{(B51)}$$

where g^* is given in equation (B28) and

$$g^{**} = -\frac{\gamma}{2}(T-\tau)\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right). \quad \text{(B52)}$$

Proof of Corollary 4: The expected jump is negative if and only $(1-G(\widehat{g}_\tau))(1-F(\widehat{g}_\tau)) > 0$, that is, if and only if

$$\left(1 - e^{\widehat{g}(T-\tau) - \frac{1}{2}(1-2\gamma)(T-\tau)^2\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)}\right)\left(1 - e^{-\gamma\widehat{g}_\tau(T-\tau) - \frac{1}{2}\gamma^2(T-\tau)^2\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)}\right) > 0 \ .$$

This condition is satisfied if and only if

$$\left(\widehat{g}_\tau - \frac{1}{2}(1-2\gamma)(T-\tau)\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)\right)\left(\widehat{g}_\tau + \frac{1}{2}\gamma(T-\tau)\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)\right) < 0 \ ,$$

which yields the claim.

QED (Corollary 4)

Corollary 5. *As risk aversion* $\gamma \to \infty$, $\mathrm{E}_\tau(J_M) \to 0$ *from above for any value of* $\widehat{g}_\tau$.

Proof of Corollary 5: The conditional expected jump is

$$E_\tau[J_M] = \mathcal{E}(\widehat{g}_\tau) = \frac{p_\tau(1-p_\tau)(1-F(\widehat{g}_\tau))(G(\widehat{g}_\tau)-1)}{p_\tau + (1-p_\tau)F(\widehat{g}_\tau)G(\widehat{g}_\tau)} \ .$$

Recall that the probability of a policy change is given by

$$p_\tau = \Pr\left(\widehat{g}_\tau < \underline{g}(c)\,|\widehat{g}_\tau\right) = \Pr\left(c < \widehat{g}_\tau(1-\gamma)(T-\tau) - \frac{(1-\gamma)^2}{2}(T-\tau)^2\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)\right) \ .$$

For $\gamma \to \infty$, we have $p_\tau \to 0$ (the term that involves γ^2 dominates), $G(\widehat{g}_\tau) \to \infty$, $F(\widehat{g}_\tau) \to 0$, and $F(\widehat{g}_\tau)G(\widehat{g}_\tau) \to 0$. In addition, an application of l'Hospital's rule gives us

$$\frac{p_\tau}{F(\widehat{g}_\tau)} \to 0 \text{ and } \frac{p_\tau}{F(\widehat{g}_\tau)G(\widehat{g}_\tau)} \to 0 \ .$$

Therefore, we have

$$\begin{aligned} E_\tau[J_M] &= \mathcal{E}(\widehat{g}_\tau) = \frac{p_\tau(1-p_\tau)(1-F)(G-1)}{p_\tau + (1-p_\tau)FG} = \frac{p_\tau(1-p_\tau)(G-1-FG+F)}{p_\tau + (1-p_\tau)FG} \\ &= \frac{\frac{p_\tau}{F}(1-p_\tau)(1-G^{-1}-F+FG^{-1})}{\frac{p_\tau}{FG} + (1-p_\tau)} \\ &\to \frac{\frac{p_\tau}{F}}{1} = 0 \ , \end{aligned}$$

where we used the fact that $F(\widehat{g}_\tau)G(\widehat{g}_\tau)^{-1} = e^{-(1+\gamma)\widehat{g}_\tau(T-\tau) + \frac{1}{2}\left(1-2\gamma-\gamma^2\right)(T-\tau)^2\left(\sigma_g^2 - \sigma_\tau^2\right)} \to 0$. Note that because both $p_\tau > 0$ and $F(\widehat{g}_\tau) > 0$, the limit of $\mathcal{E}(\widehat{g}_\tau)$ is from above, which implies that for every $\widehat{g}_\tau$, $\mathcal{E}(\widehat{g}_\tau) > 0$ for γ sufficiently large. Indeed, note that the range in which $\mathcal{E}(\widehat{g}_\tau) < 0$ is $(g^*, g^{**}) \to (-\infty, -\infty)$, so that for any given $\widehat{g}_\tau$, $\mathcal{E}(\widehat{g}_\tau)$ must turn positive as γ increases.

QED (Corollary 5)

Corollary 6. *As risk aversion* $\gamma \to 1$, $\mathrm{E}_\tau(J_M)$ *converges to a nonnegative value for any* $\widehat{g}_\tau$. *It converges to zero if and only if* $\widehat{g}_\tau = -\frac{1}{2}(T-\tau)\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)$.

Proof of Corollary 6: The conditional expected jump is

$$E_\tau[J_M] = \mathcal{E}(\widehat{g}_\tau) = \frac{p_\tau(1-p_\tau)(1-F(\widehat{g}_\tau))(G(\widehat{g}_\tau)-1)}{p_\tau + (1-p_\tau)F(\widehat{g}_\tau)G(\widehat{g}_\tau)} .$$

From the definition of p_τ,

$$p_\tau = \Pr\left(\widehat{g} < \underline{g}(c)\,|\widehat{g}\right) = \Pr\left(c < \widehat{g}_\tau(1-\gamma)(T-\tau) - \frac{(1-\gamma)^2}{2}(T-\tau)^2\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)\right) ,$$

we have that $\gamma \to 1$ implies $p_\tau \to \overline{p} = N\left(0, -\frac{1}{2}\sigma_c^2, \sigma_c^2\right)$, independent of $\widehat{g}_\tau$. In addition, as $\gamma \to 1$, we also have $F(\widehat{g}_\tau) \to 1/G(\widehat{g}_\tau)$. This implies that for every $\widehat{g}_\tau$, the expected jump converges to

$$\begin{aligned}
\mathcal{E}(\widehat{g}_\tau) &\to \overline{p}(1-\overline{p})(1-1/G(\widehat{g}_\tau))(G(\widehat{g}_\tau)-1) \\
&= \overline{p}(1-\overline{p})\left(1-e^{-\widehat{g}_\tau(T-\tau)-\frac{1}{2}(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}\right)\left(e^{\widehat{g}_\tau(T-\tau)+\frac{1}{2}(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}-1\right) \\
&= \overline{p}(1-\overline{p})\left(1-e^{-x}\right)(e^x-1) ,
\end{aligned}$$

where $x = \widehat{g}_\tau(T-\tau) + \frac{1}{2}(T-\tau)^2\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)$. This expression is always strictly positive, except for $x=0$ (i.e., $\widehat{g}_\tau = -\frac{1}{2}(T-\tau)\left(\sigma_g^2 - \widehat{\sigma}_\tau^2\right)$), in which case it is equal to zero.

QED (Corollary 6)

Corollary 7. *As risk aversion* $\gamma \to \infty$, $\mathrm{E}(J_M) \to 0$ *from above.*

Proof of Corollary 7: The proof follows from that of Corollary 5 and the result that *for every* $\widehat{g}_\tau$, we have $\mathcal{E}(\widehat{g}_\tau) \to 0$ as $\gamma \to \infty$ from above. This implies that also unconditionally

$$E[J_{M,\tau}] = \int \mathcal{E}(\widehat{g}_\tau)\,\phi\left(\widehat{g}_\tau, 0, \sigma_g^2 - \widehat{\sigma}_\tau^2\right) d\widehat{g}_\tau \to 0 ,$$

where $\phi\left(\widehat{g}_\tau, 0, \sigma_g^2 - \widehat{\sigma}_\tau^2\right)$ is the normal density with mean zero and variance $\sigma_g^2 - \widehat{\sigma}_\tau^2$. In addition, because for $\widehat{g}_\tau$, $\mathcal{E}(\widehat{g}_\tau) > 0$ for γ sufficiently large, and because $\phi\left(\widehat{g}_\tau, 0, \sigma_g^2 - \widehat{\sigma}_\tau^2\right)$ does not depend on γ, we have that $E[J_{M,\tau}]$ converges to zero from above (i.e., it is positive for a sufficiently large γ). This implies that there exists $\underline{\gamma}$ such that for $\gamma > \underline{\gamma}$, we have $E[J_{M,\tau}] > 0$.

QED (Corollary 7)

Corollary 8. *As risk aversion* $\gamma \to 1$, $\mathrm{E}(J_M)$ *converges to a positive value.*

Proof of Corollary 8: From the proof of Corollary 6, note that as $\gamma \to 1$, the unconditional expectation

$$\begin{aligned} E\left[J_{M,\tau}\right] &= E\left[E_\tau\left[J_{M,\tau}\right]\right] = E\left[\mathcal{E}\left(\widehat{g}_\tau\right)\right] \\ &\to E\left[\overline{p}\left(1-\overline{p}\right)\left(1-e^{-\widehat{g}_\tau(T-\tau)-\frac{1}{2}(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}\right)\left(e^{\widehat{g}_\tau(T-\tau)+\frac{1}{2}(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}-1\right)\right] \\ &= \overline{p}\left(1-\overline{p}\right)E\left[e^{\widehat{g}_\tau(T-\tau)+\frac{1}{2}(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}-1-1+e^{-\widehat{g}_\tau(T-\tau)-\frac{1}{2}(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}\right] \\ &= \overline{p}\left(1-\overline{p}\right)\left(e^{(T-\tau)^2\left(\sigma_g^2-\widehat{\sigma}_\tau^2\right)}-1\right) > 0\ . \end{aligned}$$

QED (Corollary 8)

Corollary 9. *The correlation between the returns of any pair of stocks for $t > \tau$ is given by*

$$\rho_t = \frac{\left[\sigma + (T-t)\,\widehat{\sigma}_t^2\sigma^{-1}\right]^2}{\left[\sigma + (T-t)\,\widehat{\sigma}_t^2\sigma^{-1}\right]^2 + \sigma_1^2}\ . \tag{B53}$$

For $t < \tau$, the correlation is given by (Corollary A3 in the paper)

$$\rho_t = \frac{\sigma_{M,t}^2}{\sigma_1^2 + \sigma_{M,t}^2}\ . \tag{B54}$$

For $t = \tau$, the instantaneous correlation is one.

Proof of Corollary 9: The statement follows from the definition of correlation, which in continuous time is given by

$$\rho_t = \frac{\mathrm{E}_t\left[\left(\frac{dM_t^i}{M_t^i}\right)\left(\frac{dM_t^j}{M_t^j}\right)\right]}{\sqrt{\mathrm{E}_t\left[\left(\frac{dM_t^i}{M_t^i}\right)^2\right]\mathrm{E}_t\left[\left(\frac{dM_t^i}{M_t^i}\right)^2\right]}} = \frac{\sigma_{M,t}^2}{\sigma_1^2 + \sigma_{M,t}^2}\ .$$

Substituting the appropriate value for $\sigma_{M,t}$ from Proposition 7, the claim of the corollary follows.

QED (Corollary 9)

Extension: Endogenous Timing of Policy Change.

Let $V\left(\widehat{g}_t, B_t, t\right)$ denote the value function given no policy change at or before time t:

$$V\left(\widehat{g}_t, B_t, t\right) = E_t\left\{\max_{\tau>t}\left\{E_\tau\left[\frac{B_T^{1-\gamma}}{1-\gamma}|\text{No change at }\tau\right], E_\tau\left[C\frac{B_T^{1-\gamma}}{1-\gamma}|\text{Change at }\tau\right]\right\}\right\}\ . \tag{B55}$$

Proposition A3. *Let the timing of the policy change be endogenous. For every* $t \in [\tau_i, \tau_{i+1})$, *the indirect utility function* $V(\widehat{g}_t, B_t, t)$ *from equation (B55) is given by*

$$V(\widehat{g}_t, B_t, t) = B_t^{1-\gamma}\Phi(\widehat{g}_t, t) \ , \tag{B56}$$

where $\Phi(\widehat{g}_t, t)$ *satisfies the partial differential equation*

$$\begin{aligned} 0 \;=\; & \frac{\partial \Phi(\widehat{g}_t, t)}{\partial t} + \left\{(1-\gamma)(\mu + \widehat{g}_t) - \frac{1}{2}\gamma(1-\gamma)\sigma^2\right\}\Phi(\widehat{g}_t, t) \\ & + \frac{1}{2}\frac{\partial^2 \Phi(\widehat{g}_t, t)}{\partial \widehat{g}_t^2}\left(\widehat{\sigma}_t^2\right)^2\sigma^{-2} + (1-\gamma)\frac{\partial \Phi(\widehat{g}_t, t)}{\partial \widehat{g}_t}\widehat{\sigma}_t^2 \ . \end{aligned} \tag{B57}$$

The boundary conditions at time τ_i *are given by*

$$\Phi(\widehat{g}_{\tau_i-}, \tau_i-) = E_{\tau_i-}\left[\max\left\{\Phi(\widehat{g}_{\tau_i}, \tau_i), \frac{1}{1-\gamma}e^{c+(1-\gamma)\mu(T-\tau_i)+\frac{1}{2}(1-\gamma)^2\sigma_g^2(T-\tau_i)^2-\gamma(1-\gamma)\frac{\sigma^2}{2}(T-\tau_i)}\right\}\right] ,$$

where the expectation is taken with respect to c *just before the policy decision at time* τ_i. *The final condition at time* T *is* $\Phi(\widehat{g}_T, T) = \frac{1}{1-\gamma}$.

Proof of Proposition A3: We have the final condition

$$V(\widehat{g}_T, B_T, T) = \frac{B_T^{1-\gamma}}{1-\gamma} \ .$$

Let τ denote a generic τ_i. Because the policy change is irreversible, we can use the aggregate capital process

$$B_T = B_\tau e^{\left(\mu + g_\tau - \frac{\sigma^2}{2}\right)(T-\tau)+\sigma(Z_T - Z_\tau)}$$

to obtain the value of expected utility at any time τ conditional on a change at τ in closed form:

$$\overline{V}(B_\tau, c, \tau) \;=\; e^c E_\tau\left[\frac{B_T^{1-\gamma}}{1-\gamma}|Yes, \widehat{g}_\tau, B_\tau, c\right] = e^c\frac{B_\tau^{1-\gamma}}{1-\gamma}e^{(1-\gamma)\mu(T-\tau)+\sigma_g^2(T-\tau)^2-\gamma(1-\gamma)\frac{\sigma^2}{2}(T-\tau)} \tag{B58}$$

A policy change occurs at time τ if

$$\overline{V}(B_\tau, c, \tau) > V(\widehat{g}_\tau, B_\tau, \tau) \ .$$

Since no policy decisions are made between τ_i and τ_{i+1}, we have for $t \in [\tau_i, \tau_{i+1})$ that $V(\widehat{g}_t, B_t, t) = E_t\left[\max\left\{V\left(\widehat{g}_{\tau_{i+1}}, B_{\tau_{i+1}}, \tau_{i+1}\right), \overline{V}\left(B_{\tau_{i+1}}, c, \tau_{i+1}\right)\right\}\right]$. It follows that V satisfies the martingale condition $E_t[dV] = 0$, that is

$$0 = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial B_t}E_t[dB_t] + \frac{\partial V}{\partial \widehat{g}_t}E_t[d\widehat{g}_t] + \frac{1}{2}\frac{\partial^2 V}{\partial B_t^2}E_t\left[dB_t^2\right] + \frac{1}{2}\frac{\partial^2 V}{\partial \widehat{g}_t^2}E_t\left[d\widehat{g}_t^2\right] + \frac{\partial^2 V}{\partial B_t \partial \widehat{g}_t}E_t[d\widehat{g}_t dB_t]$$

with the following final condition at $\tau = \tau_{i+1}$ (we drop the subscript for notational convenience):

$$V(\widehat{g}_{\tau-}, B_{\tau-}, \tau-) = E_{\tau-}\left[\max\left\{V(\widehat{g}_\tau, B_\tau, \tau), \overline{V}(B_\tau, c, \tau)\right\}\right] \tag{B59}$$

where the expectation is taken over possible values of c. Note that because of the convexity of the max operator, we cannot take the expectation inside the parenthesis.

We conjecture that the value function is:

$$V(\widehat{g}_t, B_t, t) = B_t^{1-\gamma} \Phi(\widehat{g}_t, t) \ .$$

Taking first derivatives,

$$\begin{aligned}
\frac{\partial V}{\partial B_t} &= (1-\gamma) B_t^{-\gamma} \Phi(\widehat{g}_t, t) ; \frac{\partial^2 V}{\partial B_t^2} = -\gamma (1-\gamma) B_t^{-\gamma-1} \Phi(\widehat{g}_t, t) \\
\frac{\partial V}{\partial \widehat{g}_t} &= B_t^{1-\gamma} \frac{\partial \Phi(\widehat{g}_t, t)}{\partial \widehat{g}_t} ; \frac{\partial^2 V}{\partial \widehat{g}_t^2} = B_t^{1-\gamma} \frac{\partial^2 \Phi(\widehat{g}_t, t)}{\partial \widehat{g}_t^2} \\
\frac{\partial V}{\partial t} &= B_t^{1-\gamma} \frac{\partial \Phi(\widehat{g}_t, t)}{\partial t} ; \frac{\partial^2 V}{\partial \widehat{g}_t \partial B_t} = (1-\gamma) B_t^{-\gamma} \frac{\partial \Phi(\widehat{g}_t, t)}{\partial \widehat{g}_t} \ .
\end{aligned}$$

In addition, we have for $t < \tau$,

$$E_t[dB_t] = (\mu + \widehat{g}_t) B_t; \ E\left[dB_t^2\right] = \sigma^2 B_t^2; \ E_t[d\widehat{g}_t] = 0; \ E\left[d\widehat{g}_t^2\right] = \left(\widehat{\sigma}_t^2\right)^2 \sigma^{-2}; \ E[dB_t d\widehat{g}_t] = \widehat{\sigma}_t^2 B_t \ .$$

Substituting the derivatives of the value function and the expectations in the PDE, we obtain (B57). The final condition at τ_{i+1} follows from equation (B59).

QED (Proposition A3)

Statement from the text: Conditional on a policy change at a given time τ, the value function at that time is available in closed form:

$$\overline{V}(B_\tau, \tau, c) = \frac{B_\tau^{1-\gamma}}{1-\gamma} e^{c+\mu(1-\gamma)(T-\tau)+\sigma_g^2(T-\tau)^2-\gamma(1-\gamma)\frac{\sigma^2}{2}(T-\tau)} \ . \tag{B60}$$

Proof: See proof of Proposition A3.

QED (Statement from the text)

Proposition A4: For every $t \in [\tau_i, \tau_{i+1})$, the market-to-book ratio M/B of firm j before a policy change is given by

$$\frac{M_t^j}{B_t^j} = \frac{\Omega(\widehat{g}_t, t)}{F(\widehat{g}_t, t)} \ ,$$

where $F(\widehat{g}_t, t)$ and $\Omega(\widehat{g}_t, t)$ satisfy the ODEs

$$\begin{aligned}
0 &= \frac{\partial F(\widehat{g}_t, t)}{\partial t} + \left\{-\gamma(\mu + \widehat{g}_t) + \frac{1}{2}\gamma(\gamma+1)\sigma^2\right\} F(\widehat{g}_t, t) + \frac{1}{2}\frac{\partial^2 F(\widehat{g}_t, t)}{\partial \widehat{g}_t^2}\left(\widehat{\sigma}_t^2\right)^2 \sigma^{-2} - \gamma \frac{\partial F(\widehat{g}_t, t)}{\partial \widehat{g}} \widehat{\sigma}_t^2 \\
0 &= \frac{\partial \Omega(\widehat{g}_t, t)}{\partial t} + \left\{(1-\gamma)(\mu + \widehat{g}_t) + \left(\frac{1}{2}\gamma(\gamma-1)\right)\sigma^2\right\} \Omega(\widehat{g}_t, t) \\
&\quad + \frac{1}{2}\frac{\partial^2 \Omega(\widehat{g}_t, t)}{\partial \widehat{g}_t^2}\left(\widehat{\sigma}_t^2\right)^2 \sigma^{-2} + (1-\gamma)\frac{\partial \Omega(\widehat{g}_t, t)}{\partial \widehat{g}_t} \widehat{\sigma}_t^2
\end{aligned}$$

with the following boundary conditions for $\tau = \tau_{i+1}$:

$$\begin{aligned} F\left(\widehat{g}_{\tau-}, \tau-\right) &= p_{\tau-} e^{-\gamma\mu(T-\tau)+\frac{1}{2}\gamma^2(T-\tau)^2\widehat{\sigma}_g^2+\gamma(\gamma+1)\frac{\sigma^2}{2}(T-\tau)} + (1-p_{\tau-}) F\left(\widehat{g}_\tau, \tau\right) \\ \Omega\left(\widehat{g}_{\tau-}, \tau-\right) &= p_{\tau-} e^{(1-\gamma)\mu(T-\tau)+\frac{1}{2}(1-\gamma)^2(T-\tau)^2\sigma_g^2-(1-\gamma)\gamma\frac{\sigma^2}{2}(T-\tau)} + (1-p_{\tau-}) \Omega\left(\widehat{g}_\tau, \tau\right) \end{aligned}$$

and final conditions at T, $F\left(\widehat{g}_T, T\right) = 1$ and $\Omega\left(\widehat{g}_T, T\right) = 1$. Above, $p_{\tau-}$ is the probability of a policy change at τ, given explicitly by

$$p_{\tau-} = \mathcal{N}\left(x\left(\widehat{g}_\tau, \tau\right), -\frac{1}{2}\sigma_c^2, \sigma_c^2\right) , \tag{B61}$$

where

$$\begin{aligned} x\left(\widehat{g}_\tau, \tau\right) &= \log\left(\Phi\left(\widehat{g}_\tau, \tau\right)(1-\gamma)\right) - (1-\gamma)\mu(T-\tau) \\ &\quad -\frac{1}{2}(1-\gamma)^2\sigma_g^2(T-\tau)^2 + \gamma(1-\gamma)\frac{\sigma^2}{2}(T-\tau) . \end{aligned}$$

Proof of Proposition A4: We start by computing the probability at τ that a policy change will occur, assuming it has not occurred yet. From the proof of Proposition A3, we have:

$$\begin{aligned} p_{\tau-} &= \Pr\left(\text{Change at } \tau | \widehat{g}_\tau, \tau\right) = \Pr\left(\Phi\left(\widehat{g}_\tau, \tau\right) < \frac{1}{1-\gamma} e^{c+(1-\gamma)\mu(T-\tau)+\frac{1}{2}(1-\gamma)^2\sigma_g^2(T-\tau)^2-\gamma(1-\gamma)\frac{\sigma^2}{2}(T-\tau)}\right) \\ &= \Pr\left(\Phi\left(\widehat{g}_\tau, \tau\right)(1-\gamma) e^{-(1-\gamma)\mu(T-\tau)-\frac{1}{2}(1-\gamma)^2\sigma_g^2(T-\tau)^2+\gamma(1-\gamma)\frac{\sigma^2}{2}(T-\tau)} > e^c\right) \\ &= \Pr\left(c < \log\left(\Phi\left(\widehat{g}_\tau, \tau\right)(1-\gamma)\right) - (1-\gamma)\mu(T-\tau) - \frac{(1-\gamma)^2}{2}\sigma_g^2(T-\tau)^2 + \gamma(1-\gamma)\frac{\sigma^2}{2}(T-\tau)\right) \end{aligned}$$

which leads to (B61).

To compute asset prices, we first consider the dynamics of the stochastic discount factor. The analysis is similar to above, except that market participants do not decide whether to change policy or not. Let

$$W\left(B_t, \widehat{g}_t, t\right) = E_t\left[B_T^{-\gamma} | \tau > t\right] .$$

For $t \in [\tau_i, \tau_{i+1})$, market participants know there are no policy decisions, and therefore

$$0 = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial B_t} E_t\left[dB_t\right] + \frac{\partial W}{\partial \widehat{g}_t} E_t\left[d\widehat{g}_t\right] + \frac{1}{2}\frac{\partial^2 W}{\partial B_t^2} E_t\left[dB_t^2\right] + \frac{1}{2}\frac{\partial^2 W}{\partial \widehat{g}_t^2} E_t\left[d\widehat{g}_t^2\right] + \frac{\partial^2 W}{\partial \widehat{g}_t \partial B_t} E_t\left[dB_t d\widehat{g}_t\right] .$$

When a policy change occurs at $\tau = \tau_{i+1}$, then

$$\begin{aligned} \overline{W}\left(B_\tau, \tau\right) &= E_\tau\left[B_T^{-\gamma} | \text{Change at } \tau\right] = E_\tau\left[B_\tau^{-\gamma} e^{-\gamma(\mu+g^{\text{new}})(T-\tau)+\gamma\frac{\sigma^2}{2}(T-\tau)-\gamma\sigma(Z_T-Z_\tau)}\right] \\ &= B_\tau^{-\gamma} e^{-\gamma\mu(T-\tau)+\frac{1}{2}\gamma^2(T-\tau)^2\sigma_g^2+\gamma\frac{\sigma^2}{2}(T-\tau)+\frac{\gamma^2}{2}\sigma^2(T-\tau)} = B_\tau^{-\gamma} e^{-\gamma\mu(T-\tau)+\frac{1}{2}\gamma^2(T-\tau)^2\sigma_g^2+\gamma(\gamma+1)\frac{\sigma^2}{2}(T-\tau)} \end{aligned}$$

Thus, $W\left(B_t, \widehat{g}_t, t\right)$ for $t \in [\tau_i, \tau_{i+1})$ has the following final condition at $\tau = \tau_{i+1}$:

$$W\left(B_{\tau-}, \widehat{g}_{\tau-}, \tau-\right) = p_{\tau-}\overline{W}\left(B_\tau, \tau\right) + (1-p_{\tau-}) W\left(B_\tau, \widehat{g}_\tau, \tau\right) .$$

The final condition at T is $W\left(B_T,\widehat{g}_T,T\right)=B_T^{-\gamma}$.

We conjecture

$$W\left(B_t,\widehat{g}_t,t\right)=B_t^{-\gamma}F\left(\widehat{g}_t,t\right)\ ,$$

so that

$$\begin{aligned}\frac{\partial W}{\partial t} &= B_t^{-\gamma}\frac{\partial F\left(\widehat{g}_t,t\right)}{\partial t};\frac{\partial W}{\partial \widehat{g}_t}=B_t^{-\gamma}\frac{\partial F\left(\widehat{g}_t,t\right)}{\partial \widehat{g}_t};\frac{\partial W}{\partial B_t}=-\gamma B_t^{-\gamma-1}F\left(\widehat{g}_t,t\right);\\ \frac{\partial^2 W}{\partial B_t^2} &= \gamma\left(\gamma+1\right)B_t^{-\gamma-2}F\left(\widehat{g}_t,t\right);\frac{\partial^2 W}{\partial \widehat{g}_t^2}=B_t^{-\gamma}\frac{\partial^2 F\left(\widehat{g}_t,t\right)}{\partial \widehat{g}_t^2};\frac{\partial^2 W}{\partial B_t\partial \widehat{g}_t}=-\gamma B_t^{-\gamma-1}\frac{\partial F\left(\widehat{g}_t,t\right)}{\partial \widehat{g}_t}\ .\end{aligned}$$

Substituting these expressions and the expectations in the PDE as in the proof of Proposition A3, we obtain the first PDE in the claim of the proposition. The boundary condition also follows from the conjectured solution.

Similarly, for the numerator of the pricing formula $M_t^j=E_t[\pi_T B_T^j]/\pi_t$, we must compute

$$Q\left(B_t^j,B_t,\widehat{g}_t,t\right)=E_t\left[B_T^jB_T^{-\gamma}|\tau>t\right]\ .$$

For $t\in[\tau_i,\tau_{i+1})$, market participants know there are no policy decisions, and therefore

$$\begin{aligned}0 &= \frac{\partial Q}{\partial t}+\frac{\partial Q}{\partial B_t}E\left[dB_t\right]+\frac{\partial Q}{\partial B_t^j}E\left[dB_t^j\right]+\frac{\partial Q}{\partial \widehat{g}_t}E\left[d\widehat{g}_t\right]+\frac{1}{2}\frac{\partial^2 Q}{\partial B_t^2}E\left[dB_t^2\right]+\frac{1}{2}\frac{\partial^2 Q}{\partial\left(B_t^i\right)^2}E\left[d\left(B_t^i\right)^2\right]\\ &\quad+\frac{1}{2}\frac{\partial^2 Q}{\partial \widehat{g}_t^2}E\left[d\widehat{g}_t^2\right]+\frac{\partial^2 Q}{\partial \widehat{g}_t\partial B_t}E\left[dB_td\widehat{g}_t\right]+\frac{\partial^2 Q}{\partial \widehat{g}_t\partial B_t^j}E\left[dB_t^jd\widehat{g}_t\right]+\frac{\partial^2 Q}{\partial B_t^j\partial B_t}E\left[dB_tdB_t^j\right]\ .\end{aligned}$$

When a policy change occurs at τ, then

$$\begin{aligned}\overline{Q}\left(B_\tau^j,B_\tau,\tau\right) &= E_\tau\left[B_T^jB_T^{-\gamma}|\text{Change at }\tau\right]\\ &= E_\tau\left[B_\tau^je^{(\mu+g^{\text{new}})(T-\tau)-\frac{1}{2}\sigma^2(T-t)-\frac{1}{2}\sigma_1^2(T-t)+\sigma(Z_T-Z)+\sigma_1\left(Z_T^j-Z_t^j\right)}\times\right.\\ &\quad\left.\times B_\tau^{-\gamma}e^{-\gamma(\mu+g^{\text{new}})(T-\tau)+\gamma\frac{\sigma^2}{2}(T-\tau)-\gamma\sigma(Z_T-Z_\tau)}\right]\\ &= E_\tau\left[B_\tau^je^{-\frac{1}{2}\sigma_1^2(T-t)+\sigma_1\left(Z_T^j-Z_t^j\right)}\times B_\tau^{-\gamma}e^{(1-\gamma)(\mu+g^{\text{new}})(T-\tau)+(\gamma-1)\frac{\sigma^2}{2}(T-\tau)+(1-\gamma)\sigma(Z_T-Z_\tau)}\right]\\ &= B_\tau^jB_\tau^{-\gamma}e^{(1-\gamma)\mu(T-\tau)+\frac{1}{2}(1-\gamma)^2(T-\tau)^2\sigma_g^2+(\gamma-1)\frac{\sigma^2}{2}(T-\tau)+\frac{1}{2}(1-\gamma)^2\sigma^2(T-\tau)}\\ &= B_\tau^jB_\tau^{-\gamma}e^{(1-\gamma)\mu(T-\tau)+\frac{1}{2}(1-\gamma)^2(T-\tau)^2\sigma_g^2-(1-\gamma)\gamma\frac{\sigma^2}{2}(T-\tau)}\ .\end{aligned}$$

Thus, $Q\left(B_t,\widehat{g}_t,t\right)$ for $t\in[\tau_i,\tau_{i+1})$ has the following final condition at $\tau=\tau_{i+1}$:

$$Q\left(B_{\tau-}^j,B_{\tau-},\widehat{g}_{\tau-},\tau-\right)=p_{\tau-}\overline{Q}\left(B_\tau^j,B_\tau,\tau\right)+\left(1-p_{\tau-}\right)Q\left(B_\tau^j,B_\tau,\widehat{g}_\tau,\tau\right)\ .$$

We conjecture

$$Q\left(B_t^j,B_t,\widehat{g}_t,t\right)=B_t^jB_t^{-\gamma}\Omega\left(\widehat{g}_t,t\right)\ ,$$

so that

$$\begin{aligned}
\frac{\partial Q}{\partial t} &= B_t^j B_t^{-\gamma}\frac{\partial \Omega\left(\widehat{g}_t,t\right)}{\partial t};\ \frac{\partial Q}{\partial \widehat{g}_t}=B_t^j B_t^{-\gamma}\frac{\partial \Omega\left(\widehat{g}_t,t\right)}{\partial \widehat{g}_t};\ \frac{\partial Q}{\partial B_t}=-\gamma B_t^j B_t^{-\gamma-1}\Omega\left(\widehat{g}_t,t\right);\ \frac{\partial Q}{\partial B_t^j}=B_t^{-\gamma}\Omega\left(\widehat{g}_t,t\right)\\
\frac{\partial^2 Q}{\partial B_t^2} &= \gamma\left(\gamma+1\right)B_t^j B_t^{-\gamma-2}\Omega\left(\widehat{g}_t,t\right);\ \frac{\partial^2 Q}{\partial \widehat{g}_t^2}=B_t^j B_t^{-\gamma}\frac{\partial^2 \Omega\left(\widehat{g}_t,t\right)}{\partial \widehat{g}_t^2};\ \frac{\partial^2 Q}{\partial B_t\partial \widehat{g}_t}=-\gamma B_t^j B_t^{-\gamma-1}\frac{\partial \Omega\left(\widehat{g}_t,t\right)}{\partial \widehat{g}_t}\\
\frac{\partial^2 Q}{\partial \left(B^j\right)_t^2} &= 0;\ \frac{\partial^2 Q}{\partial B_t^j\partial B_t}=-\gamma B_t^{-\gamma-1}\Omega\left(\widehat{g}_t,t\right);\ \frac{\partial^2 Q}{\partial B_t^j\partial \widehat{g}_t}=B_t^{-\gamma}\frac{\partial \Omega\left(\widehat{g}_t,t\right)}{\partial \widehat{g}_t}\ .
\end{aligned}$$

Substituting in the PDE, we obtain

$$\begin{aligned}
0 &= B_t^j B_t^{-\gamma}\frac{\partial \Omega\left(\widehat{g}_t,t\right)}{\partial t}-\gamma B_t^j B_t^{-\gamma-1}\Omega\left(\widehat{g}_t,t\right)\left(\mu+\widehat{g}_t\right)B_t+B_t^{-\gamma}\Omega\left(\widehat{g}_t,t\right)\left(\mu+\widehat{g}_t\right)B_t^j\\
&+\frac{1}{2}\gamma\left(\gamma+1\right)B_t^j B_t^{-\gamma-2}\Omega\left(\widehat{g}_t,t\right)\sigma^2 B_t^2+\frac{1}{2}B_t^j B_t^{-\gamma}\frac{\partial^2 \Omega\left(\widehat{g}_t,t\right)}{\partial \widehat{g}_t^2}\left(\widehat{\sigma}_t^2\right)^2\sigma^{-2}\\
&-\gamma B_t^j B_t^{-\gamma-1}\frac{\partial \Omega\left(\widehat{g}_t,t\right)}{\partial \widehat{g}_t}\widehat{\sigma}_t^2 B_t+B_t^{-\gamma}\frac{\partial \Omega\left(\widehat{g}_t,t\right)}{\partial \widehat{g}_t}\widehat{\sigma}_t^2 B_t^j-\gamma B_t^{-\gamma-1}\Omega\left(\widehat{g}_t,t\right)\sigma^2 B_t^j B_t\ ,
\end{aligned}$$

where we also used

$$E_t\left[dB_t^j\right] = \left(\mu+\widehat{g}_t\right)B_t^j;\ E_t\left[dB_t^j d\widehat{g}_t\right]=\widehat{\sigma}_t^2 B_t^j;\ E_t\left[dB_t^j dB_t\right]=\sigma^2 B_t^j B_t$$

The second PDE in Proposition A4 then follows after some algebraic manipulations. The boundary condition also follows from the conjecture solution.

Extension: Investment Adjustment.

Proposition 9. *In equilibrium at time τ, a randomly-selected fraction $\alpha_\tau \in [0,1]$ of firms continue investing in their risky technologies, while the remaining firms disinvest and park their capital in the risk-free technology. The government changes its policy if and only if*

$$\widehat{g}_\tau < \underline{g}\left(c,\alpha_\tau\right)\ , \tag{B62}$$

where for given (c,α_τ) the threshold $\underline{g}\left(c,\alpha_\tau\right)$ satisfies the equation

$$e^c E_\tau\left[\left[\alpha_\tau e^{\varepsilon^{yes}(\tau,T)}+(1-\alpha_\tau)\right]^{1-\gamma}\right]=E_\tau\left[\left[\alpha_\tau e^{\varepsilon^{g}(\tau,T)}+(1-\alpha_\tau)\right]^{1-\gamma}\right] \tag{B63}$$

where

$$\varepsilon^{yes} \sim N\left(\left(\mu-\frac{\sigma^2}{2}\right)(T-\tau),\sigma_g^2(T-\tau)^2+\sigma^2(T-\tau)\right) \tag{B64}$$

$$\varepsilon^{g} \sim N\left(\left(\mu+\underline{g}(c,\alpha_\tau)-\frac{\sigma^2}{2}\right)(T-\tau),\widehat{\sigma}_\tau^2(T-\tau)^2+\sigma^2(T-\tau)\right) \tag{B65}$$

and the equilibrium value of α_τ is described below.

We prove Proposition 9 together with the following statement from the text of the paper:

Statement from the text: If firm i decides at time τ to remain invested in its risky technology, its market value right after time τ is given by

$$M^i_{\tau+} = \begin{cases} M^{i,yes}_{\tau+} = B^i_{\tau+} \dfrac{\mathrm{E}_{\tau+}\left[e^{\varepsilon^{yes}(\tau,T)}\left[\alpha_\tau e^{\varepsilon^{yes}(\tau,T)}+(1-\alpha_\tau)\right]^{-\gamma}\right]}{\mathrm{E}_{\tau+}\left[\left[\alpha_\tau e^{\varepsilon^{yes}(\tau,T)}+(1-\alpha_\tau)\right]^{-\gamma}\right]} & \text{if policy changes} \\ M^{i,no}_{\tau+} = B^i_{\tau+} \dfrac{\mathrm{E}_{\tau+}\left[e^{\varepsilon^{no}(\tau,T)}\left[\alpha_\tau e^{\varepsilon^{no}(\tau,T)}+(1-\alpha_\tau)\right]^{-\gamma}\right]}{\mathrm{E}_{\tau+}\left[\left[\alpha_\tau e^{\varepsilon^{no}(\tau,T)}+(1-\alpha_\tau)\right]^{-\gamma}\right]} & \text{if policy does not change} \end{cases} \tag{B66}$$

Right before time τ, the market value of firm i is given by a weighted average of $M^{i,yes}_{\tau+}$ and $M^{i,no}_{\tau+}$ as in equation (B21), except that the weights ω_τ are given by

$$\omega_\tau = \frac{p_\tau}{p_\tau + (1-p_\tau)H_\tau} , \tag{B67}$$

where

$$H_\tau = \frac{E_\tau\left\{\left[\alpha_\tau e^{\varepsilon^{no}(\tau,T)} + (1-\alpha_\tau)\right]^{-\gamma}\right\}}{E_\tau\left\{\left[\alpha_\tau e^{\varepsilon^{yes}(\tau,T)} + (1-\alpha_\tau)\right]^{-\gamma}\right\}}$$

and p_τ is the probability of a policy change as perceived just before time τ (i.e., the probability that the condition (B62) holds).

The condition for equilibrium with $0 < \alpha_\tau < 1$:[1]

$$1 = \omega_\tau \left(\frac{M^i_{\tau+}}{B^i_{\tau+}}\right)^{yes} + (1-\omega_\tau)\left(\frac{M^i_{\tau+}}{B^i_{\tau+}}\right)^{no} \tag{B68}$$

Proof of Proposition 9: We proceed in a few steps.

Lemma A6. Let α_τ be the fraction of firms, chosen at random, that at τ choose the investment in the risky technology. In this case, aggregate capital at T is

$$\frac{B_T}{B_\tau} = \alpha_\tau e^{\left(\mu+g-\frac{\sigma^2}{2}\right)(T-\tau)+\sigma(Z_T-Z_\tau)} + (1-\alpha_\tau) , \tag{B69}$$

where $g = g^{new}$ if a policy change occurs at τ, and $g = g^{old}$ otherwise.

Proof of Lemma A6: Without loss of generality, let the firms randomly choosing the risky technology fall in the interval $[0, \alpha_\tau]$. The capital evolution of each individual firm is given by (B9). If a firm chooses the riskless technology instead, then $B^i_T = B^i_\tau$. It follows that aggregate capital at time T is

$$\begin{aligned} B_T &= \int_0^1 B^i_T di = \int_0^{\alpha_\tau} B^i_\tau e^{(\mu+g)(T-\tau)-\frac{\sigma^2}{2}(T-\tau)+\sigma(Z_T-Z_\tau)-\frac{\sigma_1^2}{2}(T-\tau)+\sigma_1\left(Z^i_T-Z^i_\tau\right)} di + \int_{\alpha_\tau}^1 B^i_\tau di \\ &= e^{(\mu+g)(T-\tau)-\frac{\sigma^2}{2}(T-\tau)+\sigma(Z_T-Z_\tau)-\frac{\sigma_1^2}{2}(T-\tau)} \int_0^{\alpha_\tau} B^i_\tau e^{\sigma_1\left(Z^i_T-Z^i_\tau\right)} di + \int_{\alpha_\tau}^1 B^i_\tau di \end{aligned}$$

[1] The same condition is obtained as a first-order condition in an alternative formulation of the problem in which a social planner chooses α_τ to maximize the investors' expected utility. See Lemma A8 below.

where $g = g^{old}$ if there is no policy change at time τ, but $g = g^{new}$ if there is one. Applying the law of large numbers, we obtain

$$\begin{aligned}\int_0^{\alpha_\tau} B_\tau^i e^{\sigma_1\left(Z_T^i - Z_\tau^i\right)} di &= \alpha_\tau \left[\frac{1}{\alpha_\tau}\int_0^{\alpha_\tau} B_\tau^i e^{\sigma_1\left(Z_T^i - Z_\tau^i\right)} di\right] = \alpha_\tau E\left[B_\tau^i e^{\sigma_1\left(Z_T^i - Z_\tau^i\right)}\right] \\ &= \alpha_\tau E\left[B_\tau^i\right] E\left[e^{\sigma_1\left(Z_T^i - Z_\tau^i\right)}\right] = \alpha_\tau E\left[B_\tau^i\right] e^{\frac{1}{2}\sigma_1^2(T-\tau)} .\end{aligned}$$

Similarly,

$$\int_{\alpha_\tau}^1 B_\tau^i di = (1-\alpha_\tau)\frac{1}{1-\alpha_\tau}\int_{\alpha_\tau}^1 B_\tau^i di = (1-\alpha_\tau) E\left[B_\tau^i\right] .$$

Substituting, we obtain

$$B_T = E\left[B_\tau^i\right]\left[\alpha_\tau e^{(\mu+g)(T-\tau)-\frac{\sigma^2}{2}(T-\tau)+\sigma(Z_T-Z_\tau)} + (1-\alpha_\tau)\right] .$$

Using the same logic, the law of large numbers also implies that $\mathrm{E}[B_\tau^i] = B_\tau$, concluding the proof of Lemma A6.

Q.E.D. (Lemma A6)

Lemma A7: The government changes its policy if and only if

$$\widehat{g}_\tau < \underline{g}(c, \alpha_\tau) , \tag{B70}$$

where the threshold $\underline{g}(c, \alpha_\tau)$ is in the claim of Proposition 9.

Proof of Lemma A7. Given the equilibrium level of firms α_τ, the government changes policy if and only if

$$E\left[\frac{CB_T^{1-\gamma}}{1-\gamma}|\text{Policy Change}\right] > E\left[\frac{B_T^{1-\gamma}}{1-\gamma}|\text{No Policy Change}\right] . \tag{B71}$$

Given the aggregate capital in Lemma A6, from the perspective of all agents in the economy, the exponent in the first term on the right-hand side of (B69) has the following distribution:

$$\left(\mu + g - \frac{\sigma^2}{2}\right)(T-\tau) + \sigma(Z_T - Z_\tau) =$$

$$= \begin{cases} \varepsilon^{no}(\tau,T) \sim N\left((\mu+\widehat{g}_\tau - \frac{\sigma^2}{2})(T-\tau), \widehat{\sigma}_\tau^2(T-\tau)^2 + \sigma^2(T-\tau)\right) & \text{if no policy change} \\ \varepsilon^{yes}(\tau,T) \sim N\left((\mu - \frac{\sigma^2}{2})(T-\tau), \sigma_g^2(T-\tau)^2 + \sigma^2(T-\tau)\right) & \text{if policy change} \end{cases}$$

Using this distribution, condition (B71) is then the same as

$$e^c E_\tau\left[\left[\alpha_\tau e^{\varepsilon^{yes}(\tau,T)} + (1-\alpha_\tau)\right]^{1-\gamma}\right] < E_\tau\left[\left[\alpha_\tau e^{\varepsilon^{no}(\tau,T)} + (1-\alpha_\tau)\right]^{1-\gamma}\right] \tag{B72}$$

where

$$\varepsilon^i(\tau,T) \sim N\left(\mu_\varepsilon(g_i, T-\tau), \sigma_\varepsilon\left(\sigma_i^2, T-\tau\right)^2\right) \tag{B73}$$

with

$$\mu_{\varepsilon}\left(g_i, T-\tau\right) = \left(\mu + g_i - \frac{\sigma^2}{2}\right)(T-\tau); \tag{B74}$$

$$\sigma_{\varepsilon}\left(\sigma_i^2, T-\tau\right)^2 = (T-\tau)^2 \sigma_i^2 + \sigma^2 (T-\tau) \tag{B75}$$

for $i = yes, no$ and $g_{no} = \widehat{g}_\tau$, $\sigma_{no}^2 = \widehat{\sigma}_\tau^2$, $g_{yes} = 0$, $\sigma_{yes}^2 = \sigma_g^2$.

For given (c, α_τ), with $\alpha_\tau > 0$, condition (B72) determines a cutoff rule, as the left-hand-side is a constant, while the right-hand-side decreases with $\widehat{g}_\tau$. Let $\underline{g}(c, \alpha_\tau)$ be the threshold that solves the equation

$$e^c E_\tau\left[\left[\alpha_\tau e^{\varepsilon^{yes}(\tau,T)} + (1-\alpha_\tau)\right]^{1-\gamma}\right] = E_\tau\left[\left[\alpha_\tau e^{\varepsilon^{g}(\tau,T)} + (1-\alpha_\tau)\right]^{1-\gamma}\right]$$

where $\varepsilon^g(\tau, T)$ is as in (B73) but with $g^i = \underline{g}(c, \alpha_\tau)$ and $\sigma_i^2 = \sigma_{no}^2$. We then have that (B72) is satisfied if and only if $\widehat{g}_\tau < \underline{g}(c, \alpha_\tau)$.

Q.E.D (Lemma A7)

Turning to prices, we first obtain the state price density at $\tau+$, an instant after the policy decision has been made. Given the aggregate capital in Lemma A6, we have

$$\pi_{\tau+} = \begin{cases} \pi_{\tau+}^{yes} = B_{\tau+}^{-\gamma} E_{\tau+}\left[\left(\alpha_\tau e^{\varepsilon^{yes}(\tau,T)} + 1 - \alpha_\tau\right)^{-\gamma}\right] & \text{if policy change} \\ \pi_{\tau+}^{no} = B_{\tau+}^{-\gamma} E_{\tau+}\left[\left(\alpha_\tau e^{\varepsilon^{no}(\tau,T)} + 1 - \alpha_\tau\right)^{-\gamma}\right] & \text{if no policy change} \end{cases} \tag{B76}$$

where $\varepsilon^i(\tau, T)$ are given in (B73).

The value of firm i that chooses the risky technology is given by $M_{\tau+}^i = E_{\tau+}\left[B_T^i \pi_T\right]/\pi_{\tau+}$. The numerator is equal to

$$\begin{aligned} M_{\tau+}^i \pi_{\tau+} &= E_{\tau+}\left[\pi_T B_T^i\right] \\ &= E_{\tau+}\left[\begin{array}{c} B_{\tau+}^{-\gamma}\left[\alpha_\tau e^{(\mu+g)(T-\tau) - \frac{\sigma^2}{2}(T-\tau) + \sigma\left(\widehat{Z}_T - \widehat{Z}_\tau\right)} + (1-\alpha_\tau)\right]^{-\gamma} \times \\ \times B_{\tau+}^i e^{(\mu+g)(T-\tau) - \frac{\sigma^2}{2}(T-\tau) + \sigma\left(\widehat{Z}_T - \widehat{Z}_\tau\right) - \frac{\sigma_1^2}{2}(T-\tau) + \sigma_1\left(Z_T^i - Z_\tau^i\right)} \end{array}\right]. \end{aligned}$$

Because $(Z_T^i - Z_\tau^i)$ is independent of all the other random variables, upon taking expectation, we obtain

$$E_{\tau+}\left[\pi_T B_T^i\right] = \begin{cases} B_{\tau+}^{-\gamma} B_{\tau+}^i \, \mathrm{E}_{\tau+}\left[e^{\varepsilon^{yes}(\tau,T)}\left[\alpha_\tau e^{\varepsilon^{yes}(\tau,T)} + (1-\alpha_\tau)\right]^{-\gamma}\right] & \text{if policy changes} \\ B_{\tau+}^{-\gamma} B_{\tau+}^i \, \mathrm{E}_{\tau+}\left[e^{\varepsilon^{no}(\tau,T)}\left[\alpha_\tau e^{\varepsilon^{no}(\tau,T)} + (1-\alpha_\tau)\right]^{-\gamma}\right] & \text{if policy does not change} \end{cases} \tag{B77}$$

where $\varepsilon^{yes}(\tau, T)$ and $\varepsilon^{no}(\tau, T)$ are defined in (B73). Equation (B66) follows from (B77) and (B76).

Finally, we compute market values at time τ right before the policy decision is taken. Given the equilibrium function $\underline{g}(c, \alpha_\tau)$, let the probability of a policy change be

$$p_\tau = \Pr\left(\widehat{g}_\tau < \underline{g}(c, \alpha_\tau) | \widehat{g}_\tau, \alpha_\tau\right) = \int_{c:\widehat{g}_\tau < \underline{g}(c,\alpha_\tau)} \phi_c\left(c; -\frac{\sigma_c^2}{2}, \sigma_c^2\right) dc\ .$$

Using the law of iterated expectations, we obtain

$$\pi_\tau = p_\tau \pi_{\tau+}^{yes} + (1-p_\tau)\pi_{\tau+}^{no} \tag{B78}$$

$$E_\tau\left[\pi_T B_T^i\right] = p_\tau E_{\tau+}\left[\pi_T B_T^i | yes\right] + (1-p_\tau) E_{\tau+}\left[\pi_T B_T^i | no\right] \ , \tag{B79}$$

where $E_{\tau+}\left[\pi_T B_T^i | yes\right]$ and $E_{\tau+}\left[\pi_T B_T^i | no\right]$ are given in (B77). We then obtain the market value right before the decision at time τ:

$$\begin{aligned} M_\tau^i &= \frac{E_\tau\left[\pi_T B_T^i\right]}{\pi_\tau} = \frac{p_\tau E_{\tau+}\left[\pi_T B_T^i | yes\right] + (1-p_\tau) E_{\tau+}\left[\pi_T B_T^i | no\right]}{p_\tau \pi_{\tau+}^{yes} + (1-p_\tau)\pi_{\tau+}^{no}} \\ &= \frac{p_\tau \frac{E_{\tau+}\left[\pi_T B_T^i | yes\right]}{\pi_{\tau+}^{yes}} + (1-p_\tau)\frac{\pi_{\tau+}^{no}}{\pi_{\tau+}^{yes}} \frac{E_{\tau+}\left[\pi_T B_T^i | no\right]}{\pi_{\tau+}^{no}}}{p_\tau + (1-p_\tau)\frac{\pi_{\tau+}^{no}}{\pi_{\tau+}^{yes}}} \\ &= \omega M_{\tau+}^{yes} + (1-\omega) M_{\tau+}^{no} \ , \end{aligned}$$

where ω is in (B67) with $H = \frac{\pi_{\tau+}^{no}}{\pi_{\tau+}^{yes}}$.

We finally obtain the equilibrium condition for a Nash Equilibrium. A firm that chooses the riskless technology has $B_T^i = B_\tau^i$ and thus a market price $M_\tau^i = \frac{E_\tau\left[\pi_T B_T^i\right]}{\pi_\tau} = B_\tau^i \frac{E_\tau[\pi_T]}{\pi_\tau} = B_\tau^i$. Thus, a value-maximizing manager strictly prefers the risky technology if and only if $M_\tau^i > B_\tau^i$. For there to be an equilibrium with a fraction α_τ of firms chosing the risky technology, we must have the indifference condition $\frac{M_\tau^i}{B_\tau^i} = 1$, otherwise all firms would opt for either the risky or the riskless technology. From the pricing formula of stocks (B66), and because $B_{\tau+} = B_\tau$ (capital does not jump at the announcement), this indifference condition implies condition (B68).

To summarize, the Nash equilibrium is as follows: Given $\widehat{g}_\tau$, let there be α_τ such that condition (B68) is satisfied. Given that the fraction α_τ of firms choose the risky technology, let $\underline{g}(c, \alpha_\tau)$ be the solution to equation (B63). The government changes its policy if and only if $\widehat{g}_\tau < \underline{g}(c, \alpha_\tau)$. Given this policy function of the government, the equilibrium α_τ must be consistent with $p_\tau = \Pr\left(\widehat{g}_\tau < \underline{g}(c, \alpha_\tau)\right)$, which affects M_τ^i / B_τ^i and thus also the equilibrium condition (B68).

QED (Proposition 9)

Lemma A8: The equilibrium condition (B68), obtained in a setting in which firms maximize their market values, is equivalent to a first-order condition in an alternative setting in which a social planner chooses α_τ to maximize the investors' expected utility.

Proof of Lemma A8. Consider a social planner/representative manager who can choose at time τ how much investment in physical capital to make. Specifically, this manager chooses the fraction α_τ of firms that will invest in the risky technology. Physical investments are irreversible and the decision is made without knowing whether the government will change its policy (as in the paper). The representative manager decides according to

$$\max_{\alpha_\tau} E_\tau\left(\frac{B_T^{1-\gamma}}{1-\gamma}\right) = \max_{\alpha_\tau} \ p_\tau E_\tau\left(\frac{B_T^{1-\gamma}}{1-\gamma} | yes\right) + (1-p_\tau) E_\tau\left(\frac{B_T^{1-\gamma}}{1-\gamma} | no\right) \ ,$$

where

$$\begin{aligned} B_T|_{yes} &= B_\tau \left(\alpha_\tau e^{\varepsilon^{yes}(t,T)} + (1-\alpha_\tau)\right) \\ B_T|_{no} &= B_\tau \left(\alpha_\tau e^{\varepsilon^{no}(t,T)} + (1-\alpha_\tau)\right) \end{aligned}$$

and where $\varepsilon^{yes}(t,T)$ and $\varepsilon^{no}(t,T)$ are defined in the paper. Substitute

$$\max_{\alpha_\tau} E_\tau\left(\frac{B_T^{1-\gamma}}{1-\gamma}\right) = \frac{B_\tau^{1-\gamma}}{1-\gamma}\left\{\begin{array}{c} p_\tau E_\tau\left(\left(\alpha_\tau e^{\varepsilon^{yes}(t,T)} + (1-\alpha_\tau)\right)^{1-\gamma}\right) \\ +(1-p_\tau) E_\tau\left(\left(\alpha_\tau e^{\varepsilon^{no}(t,T)} + (1-\alpha_\tau)\right)^{1-\gamma}\right) \end{array}\right\}.$$

Assuming an interior solution $0 < \alpha_\tau < 1$, the FOC with respect to α_τ is

$$\begin{aligned} 0 &= p_\tau E_\tau\left[\left(\alpha_\tau e^{\varepsilon^{yes}(t,T)} + (1-\alpha_\tau)\right)^{-\gamma}\left(e^{\varepsilon^{yes}(t,T)} - 1\right)\right] \\ &\quad + (1-p_\tau) E_\tau\left[\left(\alpha_\tau e^{\varepsilon^{no}(t,T)} + (1-\alpha_\tau)\right)^{-\gamma}\left(e^{\varepsilon^{no}(t,T)} - 1\right)\right] \end{aligned}$$

or, equivalently,

$$\begin{aligned} & p_\tau E_\tau\left[\left(\alpha_\tau e^{\varepsilon^{yes}(t,T)} + (1-\alpha_\tau)\right)^{-\gamma}\right] + (1-p_\tau) E_\tau\left[\left(\alpha_\tau e^{\varepsilon^{no}(t,T)} + (1-\alpha_\tau)\right)^{-\gamma}\right] \\ = \; & p_\tau E_\tau\left[\left(\alpha_\tau e^{\varepsilon^{yes}(t,T)} + (1-\alpha_\tau)\right)^{-\gamma} e^{\varepsilon^{yes}(t,T)}\right] + (1-p_\tau) E_\tau\left[\left(\alpha_\tau e^{\varepsilon^{no}(t,T)} + (1-\alpha_\tau)\right)^{-\gamma} e^{\varepsilon^{no}(t,T)}\right] \end{aligned}$$

or, equivalently,

$$1 = \frac{p_\tau E_\tau\left[\left(\alpha_\tau e^{\varepsilon^{yes}(t,T)} + (1-\alpha_\tau)\right)^{-\gamma} e^{\varepsilon^{yes}(t,T)}\right] + (1-p_\tau) E_\tau\left[\left(\alpha_\tau e^{\varepsilon^{no}(t,T)} + (1-\alpha_\tau)\right)^{-\gamma} e^{\varepsilon^{no}(t,T)}\right]}{p_\tau E_\tau\left[\left(\alpha_\tau e^{\varepsilon^{yes}(t,T)} + (1-\alpha_\tau)\right)^{-\gamma}\right] + (1-p_\tau) E_\tau\left[\left(\alpha_\tau e^{\varepsilon^{no}(t,T)} + (1-\alpha_\tau)\right)^{-\gamma}\right]}.$$

This is the same condition as in the paper, where α_τ is chosen as a Nash equilibrium in which all firms have $\frac{M_\tau^i}{B_\tau^i} = 1$.

QED (Lemma A8)

Extension: Heterogeneous Betas.

Lemma A9 (Learning): Let $g \sim N(0, \sigma_g^2)$, let β denote the $N \times 1$ vector of government betas, and let $d\mathbf{s}_t$ denote the $N \times 1$ vector of signals that investors receive at t, given by

$$d\mathbf{s}_t = (\mu \mathbf{1}_N + \beta g)\, dt + \sigma d\mathbf{Z}_t$$

Then, for $t \in [0, \tau]$ the posterior is $g \sim N(\widehat{g}_t, \widehat{\sigma}_t^2)$, with

$$d\widehat{g}_t = \widehat{\sigma}_t^2 \sigma^{-1} \beta' d\widehat{\mathbf{Z}}_t \quad \text{and} \quad \widehat{\sigma}_t^2 = \frac{1}{\frac{1}{\sigma_g^2} + \left(\frac{\beta'\beta}{\sigma^2}\right) t}$$

Above, $d\widehat{\mathbf{Z}}_t$ is a $N \times 1$ vector of Brownian motions defined by

$$d\widehat{\mathbf{Z}}_t = \frac{1}{\sigma}\left[d\mathbf{s}_t - E_t\left(d\mathbf{s}_t\right)\right]$$

The profitability process for firm i in sector n under the filtered probability measure is then

$$d\Pi_t^i = \left(\mu + \beta^n \widehat{g}_t\right) dt + \sigma d\widehat{Z}_{n,t} + \sigma_1 dZ_{it} \tag{B80}$$

Proof of Lemma A9. The proof of the learning dynamics follows from an application of the Kalman Bucy filter (see e.g. Liptser and Shiryaev (1977)). From the definition of the new Brownian motions $d\widehat{\mathbf{Z}}_t$, we can write the signals under the filtered probability measure:

$$d\mathbf{s}_t = \left(\mu \mathbf{1}_N + \beta \widehat{g}_t\right) dt + \sigma d\widehat{\mathbf{Z}}_t$$

This implies that for every $n = 1, \ldots, N$, we have the following identity:

$$\beta^n g dt + \sigma dZ_{nt} = \beta^n \widehat{g}_t dt + \sigma d\widehat{Z}_{nt} \tag{B81}$$

The dynamics of the profitability process under the filtered measure follow immediately.

QED (Lemma A9)

Lemma A10. Aggregate capital at T is given by

$$\frac{B_T}{B_\tau} = e^{\mu(T-\tau) - \frac{\sigma^2}{2}(T-\tau)} \sum_{n=1}^{N} w_\tau^n e^{\beta^n g(T-\tau) + \sigma(Z_{nT} - Z_{n\tau})}$$

where

$$w_\tau^n = \frac{B_\tau^n}{B_\tau}$$

Proof of Lemma A10. There are N sectors of firms, each with mass λ^n, where $\sum_{n=1}^{N} \lambda^n = 1$. Denote by $\Lambda^n \subset [0,1]$ the set of firms in sector n. The aggregate capital at T is then

$$B_T = \int B_T^i di = \sum_{n=1}^{N} \int_{\Lambda^n} B_T^i di = \sum_{n=1}^{N} B_T^n$$

where B_T^n is the aggregate capital in sector n. Using the same steps as in the proof of Lemma A6, we can express the latter as follows:

$$\begin{aligned} B_T^n &= \int_{\Lambda^n} B_\tau^i e^{(\mu+\beta^n g)(T-\tau) - \frac{\sigma^2}{2}(T-\tau) + \sigma\left(Z_{n,T} - Z_{n,\tau}\right) - \frac{\sigma_1^2}{2}(T-\tau) + \sigma_1\left(Z_{i,T} - Z_{i,\tau}\right)} di \\ &= e^{(\mu+\beta^n g)(T-\tau) - \frac{\sigma^2}{2}(T-\tau) + \sigma(Z_{nT} - Z_{n\tau}) - \frac{\sigma_1^2}{2}(T-\tau)} \int_{\Lambda^n} B_\tau^i e^{\sigma_1\left(Z_{i,T} - Z_{i,\tau}\right)} di \\ &= B_\tau^n e^{(\mu+\beta^n g)(T-t) - \frac{\sigma^2}{2}(T-t) + \sigma(Z_{nT} - Z_{nt})} \end{aligned} \tag{B82}$$

Thus

$$B_T = \sum_{n=1}^{N} B_T^n = \sum_{n=1}^{N} B_\tau^n e^{(\mu+\beta^n g)(T-\tau)-\frac{\sigma^2}{2}(T-\tau)+\sigma(Z_{nT}-Z_{n\tau})}$$

or

$$\frac{B_T}{B_\tau} = e^{\mu(T-\tau)-\frac{\sigma^2}{2}(T-\tau)} \sum_{n=1}^{N} w_\tau^n e^{\beta^n g(T-\tau)+\sigma(Z_{nT}-Z_{n\tau})} \tag{B83}$$

where

$$w_\tau^n = \frac{B_\tau^n}{B_\tau}$$

For later reference, note that the vector

$$\beta (T-\tau) g + \sigma (\mathbf{Z}_T - \mathbf{Z}_\tau) \sim N\left(\beta (T-\tau) \widehat{g}_\tau, \widehat{\sigma}_\tau^2 (T-\tau)^2 \beta\beta' + \sigma^2 (T-\tau)\right) \tag{B84}$$

QED (Lemma A10)

Proposition 10. *The government changes its policy at time τ if and only if*

$$\widehat{g}_\tau \;<\; \underline{g}(c, \mathbf{w}_\tau) \;, \tag{B85}$$

where the threshold $\underline{g}(c, \mathbf{w}_\tau)$ is the solution to

$$e^c \int_{\mathbb{R}^n} \left(\sum_{n=1}^{N} w_\tau^n e^{x^n}\right)^{1-\gamma} \phi\left(\mathbf{x}|0, \sigma_g^2, \tau\right) d\mathbf{x} = \int_{\mathbb{R}^n} \left(\sum_{n=1}^{N} w_\tau^n e^{x^n}\right)^{1-\gamma} \phi\left(\mathbf{x}|\underline{g}(c, \mathbf{w}), \widehat{\sigma}_\tau^2, \tau\right) d\mathbf{x}$$

and $\phi(\mathbf{x}|a, b, t)$ denotes the multivariate normal density

$$N\left(a \ (T-t) \beta, b (T-t)^2 \beta\beta' + \sigma^2 (T-t)\right) \tag{B86}$$

Proof of Proposition 10. The government chooses a new policy if and only if

$$E_\tau\left[\frac{C B_T^{1-\gamma}}{1-\gamma} | yes\right] > E_\tau\left[\frac{B_T^{1-\gamma}}{1-\gamma} | no\right]$$

Substituting aggregate capital in (B83), we obtain the condition

$$e^c E_\tau\left[\left(\sum_{n=1}^{N} w_t^n e^{\beta^n g(T-\tau)+\sigma(Z_{nT}-Z_{n\tau})}\right)^{1-\gamma} | yes\right] < E_\tau\left[\left(\sum_{n=1}^{N} w_t^n e^{\beta^n g(T-\tau)+\sigma(Z_{nT}-Z_{n\tau})}\right)^{1-\gamma} | no\right]$$

Using (B84), we can express the expectation as

$$e^c \int_{\mathbb{R}^n} \left(\sum_{n=1}^{N} w_\tau^n e^{x^n}\right)^{1-\gamma} \phi\left(\mathbf{x}|0, \sigma_g^2, \tau\right) d\mathbf{x} < \int_{\mathbb{R}^n} \left(\sum_{n=1}^{N} w_\tau^n e^{x^n}\right)^{1-\gamma} \phi\left(\mathbf{x}|\widehat{g}_\tau, \widehat{\sigma}_\tau^2, \tau\right) d\mathbf{x} \tag{B87}$$

where $\phi(\mathbf{x}|a,b,t)$ denotes the multivariate normal distribution as in (B86). Given $\mathbf{w}_\tau$, the left-hand side of (B87) is independent of $\widehat{g}_\tau$. Assuming $\beta > \mathbf{0}$, the right-hand side is monotonically decreasing in $\widehat{g}_\tau$ (as the mean of each x^n increases, but the variance covariance matrix is independent of $\widehat{g}_\tau$). It follows that there is a cutoff $\underline{g}(\mathbf{w}_\tau, c)$ such that a change in policy occurs if and only if

$$\widehat{g}_\tau < \underline{g}(\mathbf{w}_\tau, c)$$

QED (Proposition 10)

Proposition A5. (a) For $t \geq \tau+$, the stochastic discount factor is given by[2]

$$\pi_t = B_t^{-\gamma} e^{-\gamma\mu(T-t)+\gamma\frac{\sigma^2}{2}(T-t)} \Omega\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right) \tag{B88}$$

where

$$\Omega\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right) = \int_{\mathbb{R}^N} \left(\sum_n w_t^n e^{x^n}\right)^{-\gamma} \phi\left(\mathbf{x}|\widehat{g}_t, \widehat{\sigma}_t^2, t\right) d\mathbf{x}$$

and $\phi(\mathbf{x}|\widehat{g}_t, \widehat{\sigma}_t^2)$ is the multivariate normal distribution as in (B86).

(b) At τ, right before the announcement, the stochastic discount factor is given by

$$\pi_\tau = B_\tau^{-\gamma} e^{-\gamma\mu(T-\tau)+\gamma\frac{\sigma^2}{2}(T-\tau)} \left[p_\tau \Omega\left(\mathbf{w}_\tau, 0, \sigma_g^2, \tau+\right) + (1-p_\tau)\Omega\left(\mathbf{w}_\tau, \widehat{g}_\tau, \widehat{\sigma}_\tau^2, \tau+\right)\right] \tag{B89}$$

where p_τ is the probability of a policy change at τ, $p_\tau = Pr\left(\widehat{g}_\tau < \underline{g}(w_\tau, c)\right)$. This probability can also be expressed as

$$p_\tau = \mathcal{N}\left(\underline{c}\left(\mathbf{w}_\tau, \widehat{g}_\tau, \widehat{\sigma}_\tau^2\right), -\frac{1}{2}\sigma_c^2, \sigma_c^2\right) \tag{B90}$$

where $\mathcal{N}(., b, c)$ denotes the cumulative normal distribution with mean b and variance c, and

$$\underline{c}\left(\mathbf{w}_\tau, \widehat{g}_\tau, \widehat{\sigma}_\tau^2\right) = \log\left(\frac{\int_{\mathbb{R}^N} \left(\sum w_\tau^n e^{x^n}\right)^{1-\gamma} \phi\left(\mathbf{x}|\widehat{g}_\tau, \widehat{\sigma}_\tau^2, \tau\right) d\mathbf{x}}{\int_{\mathbb{R}^N} \left(\sum w_\tau^n e^{x^n}\right)^{1-\gamma} \phi\left(\mathbf{x}|0, \sigma_g^2, \tau\right) d\mathbf{x}}\right)$$

(c) For $t < \tau$, the stochastic discount factor can be computed as

$$\pi_t = B_t^{-\gamma} e^{-\gamma\mu(T-\tau)+\gamma\frac{\sigma^2}{2}(T-\tau)} \Omega\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right)$$

where, with slight abuse of notation, we denote for $t < \tau$

$$\Omega\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right) = E_t\left\{\left(\frac{B_\tau}{B_t}\right)^{-\gamma} \left[p_\tau \Omega\left(\mathbf{w}_\tau, 0, \sigma_g^2, \tau+\right) + (1-p_\tau)\Omega\left(\mathbf{w}_\tau, \widehat{g}_\tau, \widehat{\sigma}_\tau^2, \tau+\right)\right]\right\}$$

(d) The dynamics of the SDF are given by

$$\frac{d\pi_t}{\pi_t} = -\sigma'_{\pi,t} d\widehat{\mathbf{Z}}_t$$

[2]For notational convenience, we suppress the dependence of the SDF on a constant Lagrange multiplier.

where

$$\sigma_{\pi,t}=\gamma\sigma\mathbf{w}_t-\sigma\left(\mathbf{w}_t\odot\left[\boldsymbol{\Omega}_w-\boldsymbol{\Omega}_w'\mathbf{w}_t\right]\frac{1}{\Omega}\right)-\frac{\Omega_g}{\Omega}\widehat{\sigma}_t^2\sigma^{-1}\beta \tag{B91}$$

and where "$\odot$" denotes "element-by-element" multiplication.

Proof of Proposition A5. (a) The stochastic discount factor is

$$\pi_t=E_t\left[B_T^{-\gamma}\right] \;=\; B_t^{-\gamma}e^{-\gamma\mu(T-t)+\gamma\frac{\sigma^2}{2}(T-t)}E_t\left[\left(\sum_{n=1}^N w_t^n e^{\beta^n g(T-t)+\sigma(Z_{nT}-Z_{nt})}\right)^{-\gamma}\right]$$

and the claims follows from (B84).

Part (b). Expression (B89) follows from $E_\tau\left[\pi_{\tau+}\right]=E_\tau\left[\pi_{\tau+}|yes\right]p_\tau+E_\tau\left[\pi_{\tau+}|no\right](1-p_\tau)$. The cutoff $\underline{c}\left(\mathbf{w}_\tau,\widehat{g}_\tau,\widehat{\sigma}_\tau^2\right)$ immediately follows from (B87).

Part (c) follows from the condition that the SDF is a martingale (under $r=0$).

Part (d). From Ito's Lemma we have

$$\frac{d\pi_t}{\pi_t}=o\left(dt\right)-\gamma\frac{dB_t}{B_t}+\sum_n\frac{\Omega_{w^n}}{\Omega}dw_t^n+\frac{\Omega_g}{\Omega}d\widehat{g}_t$$

From $w_t^m=B_t^mB_t^{-1}$ we have

$$\begin{aligned}
dw_t^m &= B_t^{-1}dB_t^m-B_t^mB_t^{-2}dB_t+B_t^mB_t^{-3}dB_t^2-B_t^{-2}dB_tdB_t^m\\
&= w_t^m\left(\left(\mu+\beta^m\widehat{g}_t\right)dt+\sigma d\widehat{Z}_{m,t}\right)-w_t^m\left\{\left[\mu_t+\left(\sum_{n=1}^N w_t^n\beta^n\right)\widehat{g}_t\right]dt+\sigma\left(\sum_{n=1}^N w_t^n d\widehat{Z}_{n,t}\right)\right\}\\
&\quad +w_t^m\sigma^2\left(\sum_{n=1}^N\left(w_t^n\right)^2\right)dt-\sigma^2\left(w_t^m\right)^2dt
\end{aligned}$$

Above, we used the fact that from (B82) and the identity (B81), the processes for B_t^m and B_t under the filtered probability measure are:

$$\begin{aligned}
\frac{dB_t^m}{B_t^m} &= \left(\mu+\beta^m\widehat{g}_t\right)dt+\sigma d\widehat{Z}_{m,t}\\
\frac{dB_t}{B_t} &= \left\{\mu+\left(\sum_{n=1}^N w_t^n\beta^n\right)\widehat{g}_t\right\}dt+\sigma\left(\sum_{n=1}^N w_t^n d\widehat{Z}_{n,t}\right)
\end{aligned}$$

Therefore, imposing the martingale condition $E_t[d\pi_t]=0$, we obtain

$$\frac{d\pi_t}{\pi_t} \;=\; -\gamma\sigma\mathbf{w}_t^n d\widehat{\mathbf{Z}}_t+\sum_{m=1}^N\frac{\Omega_{w^m}}{\Omega}w_t^m\sigma\left(d\widehat{Z}_{m,t}-\mathbf{w}_t d\widehat{\mathbf{Z}}_t\right)+\frac{\Omega_g}{\Omega}\widehat{\sigma}_t^2\sigma^{-1}\beta' d\widehat{\mathbf{Z}}_t$$

The claim follows.

QED (Proposition A5)

Proposition A6. (a) For $t \geq \tau+$, the market-to-book ratios of the aggregate market and of firm i in sector m are:

$$\frac{M_t}{B_t} = e^{\mu(T-t)-\frac{\sigma^2}{2}(T-t)}\frac{\Phi\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right)}{\Omega\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right)} \tag{B92}$$

$$\frac{M_t^{m,i}}{B_t^{m,i}} = e^{\mu(T-t)-\frac{\sigma^2}{2}(T-t)}\frac{\Phi^m\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right)}{\Omega\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right)} \tag{B93}$$

where

$$\Phi\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right) = \int_{\mathbb{R}^N}\left(\sum_n w_t^n e^{x^n}\right)^{1-\gamma}\phi\left(\mathbf{x}|\widehat{g}_t, \widehat{\sigma}_t^2, t\right)d\mathbf{x} \tag{B94}$$

$$\Phi^m\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right) = \int_{\mathbb{R}^N}\left(\sum_{n=1}^N w_t^n e^{x^n}\right)^{-\gamma} e^{x^m}\phi\left(\mathbf{x}|\widehat{g}_t, \widehat{\sigma}_t^2, t\right)d\mathbf{x} \tag{B95}$$

(b) At τ, right before the announcement, the M/B's are

$$\frac{M_\tau}{B_\tau} = e^{\mu(T-\tau)-\frac{\sigma^2}{2}(T-\tau)}\frac{p_\tau\Phi\left(\mathbf{w}_\tau, 0, \sigma_g^2, \tau\right)+(1-p_\tau)\Phi\left(\mathbf{w}_\tau, \widehat{g}_\tau, \widehat{\sigma}_\tau^2, \tau\right)}{p_\tau\Omega\left(\mathbf{w}_\tau, 0, \sigma_g^2, \tau\right)+(1-p_\tau)\Omega\left(\mathbf{w}_\tau, \widehat{g}_\tau, \widehat{\sigma}_\tau^2, \tau\right)} \tag{B96}$$

$$\frac{M_\tau^{m,i}}{B_\tau^{m,i}} = e^{\mu(T-t)-\frac{\sigma^2}{2}(T-t)}\frac{p_\tau\Phi^m\left(\mathbf{w}_\tau, 0, \sigma_g^2, \tau\right)+(1-p_\tau)\Phi^m\left(\mathbf{w}_\tau, \widehat{g}_\tau, \widehat{\sigma}_\tau^2, \tau\right)}{p_\tau\Omega\left(\mathbf{w}_\tau, 0, \sigma_g^2, \tau\right)+(1-p_\tau)\Omega\left(\mathbf{w}_\tau, \widehat{g}_\tau, \widehat{\sigma}_\tau^2, \tau\right)} \tag{B97}$$

(c) For $t < \tau$, the M/B's are

$$\frac{M_t}{B_t} = e^{\mu(T-\tau)-\frac{\sigma^2}{2}(T-\tau)}\frac{\Phi\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right)}{\Omega\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right)} \tag{B98}$$

$$\frac{M_t^{m,i}}{B_t^{m,i}} = e^{\mu(T-\tau)-\frac{\sigma^2}{2}(T-\tau)}\frac{\Phi^m\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right)}{\Omega\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right)} \tag{B99}$$

where, with a slight abuse of notation, for $t < \tau$ we denote

$$\Phi\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right) = E_t\left\{\left(\frac{B_\tau}{B_t}\right)^{1-\gamma}\left[p_\tau\Phi\left(\mathbf{w}_\tau, 0, \sigma_g^2, \tau+\right)+(1-p_\tau)\Phi\left(\mathbf{w}_\tau, \widehat{g}_\tau, \widehat{\sigma}_\tau^2, \tau+\right)\right]\right\}$$

$$\Phi^m\left(\mathbf{w}_t, \widehat{g}_t, \widehat{\sigma}_t^2, t\right) = E_t\left\{\left(\frac{B_\tau}{B_t}\right)^{-\gamma}\frac{B_\tau^m}{B_t^m}\left[p_\tau\Phi^m\left(\mathbf{w}_\tau, 0, \sigma_g^2, \tau+\right)+(1-p_\tau)\Phi^m\left(\mathbf{w}_\tau, \widehat{g}_\tau, \widehat{\sigma}_\tau^2, \tau+\right)\right]\right\}$$

(d) Stock return processes are

$$\frac{dM_t}{M_t} = \mu_{M,t}dt + \sigma'_{M,t}d\widehat{\mathbf{Z}}_t \tag{B100}$$

$$\frac{dM_t^{m,i}}{M_t^{m,i}} = \mu_{m,t}dt + \sigma'_{m,t}d\widehat{\mathbf{Z}}_t + \sigma_1 dZ_t^i \tag{B101}$$

where

$$\sigma_{M,t} = \sigma \mathbf{w}_t + \sigma \mathbf{w}_t \odot \left[\left(\frac{\mathbf{\Phi}_w}{\Phi} - \frac{\mathbf{\Omega}_w}{\Omega} \right) - \left(\frac{\mathbf{\Phi}_w}{\Phi} - \frac{\mathbf{\Omega}_w}{\Omega} \right)' \mathbf{w}_t \right] + \left(\frac{\Phi_g}{\Phi} - \frac{\Omega_g}{\Omega} \right) \widehat{\sigma}_t^2 \sigma^{-1} \beta$$

$$\sigma_{m,t} = \sigma \iota_m + \sigma \mathbf{w}_t \odot \left[\left(\frac{\mathbf{\Phi}_w^m}{\Phi} - \frac{\mathbf{\Omega}_w}{\Omega} \right) - \left(\frac{\mathbf{\Phi}_w^m}{\Phi} - \frac{\mathbf{\Omega}_w}{\Omega} \right)' \mathbf{w}_t \right] + \left(\frac{\Phi_g^m}{\Phi} - \frac{\Omega_g^m}{\Omega} \right) \widehat{\sigma}_t^2 \sigma^{-1} \beta$$

with ι_m denoting the $m-th$ column of the identity matrix, and

$$\mu_{M,t} = \sigma'_{M,t} \sigma_{\pi,t} \quad \text{(B102)}$$

$$\mu_{m,t} = \sigma'_{m,t} \sigma_{\pi,t} \quad \text{(B103)}$$

Proof of Proposition A6. (a) Aggregate market: From the Euler equation

$$M_t \pi_t = E_t \left[B_T^{-\gamma} B_T \right] = E_t \left[B_T^{1-\gamma} \right] = B_t^{1-\gamma} e^{(1-\gamma)\mu(T-t) - (1-\gamma)\frac{\sigma^2}{2}(T-t)} E_t \left[\left(\sum_{n=1}^N w_t^n e^{\beta^n g(T-t) + \sigma(Z_{nT} - Z_{nt})} \right)^{1-\gamma} \right]$$

The claims follows from (B84), after substituting for π_t from (B88). For a firm i in sector m we have that capital at T is

$$B_T^{i,m} = B_\tau^{i,m} e^{(\mu + \beta^m g)(T-\tau) - \frac{\sigma^2}{2}(T-\tau) + \sigma\left(Z_{m,T} - Z_{m,\tau}\right) - \frac{\sigma_1^2}{2}(T-\tau) + \sigma_1\left(Z_{i,T} - Z_{i,\tau}\right)}$$

Thus, from the Euler equation

$$\begin{aligned} M_t^{i,m} \pi_t &= E_t \left[\pi_T B_T^{i,m} \right] = B_t^{-\gamma} B_t^{i,m} e^{(1-\gamma)\mu(T-t) + (\gamma-1)\frac{\sigma^2}{2}(T-t) - \frac{\sigma_1^2}{2}(T-\tau)} \\ &\quad \times E_t \left[\left(\sum_{n=1}^N w_t^n e^{\beta^n g(T-t) + \sigma(Z_{nT} - Z_{nt})} \right)^{-\gamma} e^{\beta^m g(T-t) + \sigma(Z_{mT} - Z_{mt}) + \sigma_1\left(Z_{i,T} - Z_{i,\tau}\right)} \right] \\ &= B_t^{-\gamma} B_t^{i,m} e^{(1-\gamma)\mu(T-t) + (\gamma-1)\frac{\sigma^2}{2}(T-t)} \times E_t \left[\left(\sum_{n=1}^N w_t^n e^{\beta^n g(T-t) + \sigma(Z_{nT} - Z_{nt})} \right)^{-\gamma} e^{\beta^m g(T-t) + \sigma(Z_{mT} - Z_{mt})} \right] \\ &= B_t^{-\gamma} B_t^{i,m} e^{(1-\gamma)\mu(T-t) + (\gamma-1)\frac{\sigma^2}{2}(T-t)} \int_{\mathbb{R}^N} \left(\sum_{n=1}^N w_t^n e^{x^n} \right)^{-\gamma} e^{x^m} \phi\left(\mathbf{x} | \widehat{g}_t, \widehat{\sigma}_t^2, t\right) d\mathbf{x} \end{aligned}$$

The claim follows from (B84), after substituting for π_t from (B88).

Part (b). The proof is identical to the one for the SDF (part (b) of Proposition A5).

Part (c). The proof is identical to the one for the SDF (part (c) of Proposition A5).

Part (d). Consider the aggregate market. From Ito's Lemma, we have

$$\begin{aligned} \frac{dM_t}{M_t} &= o(dt) + \frac{dB_t}{B_t} + \sum_{m=1}^N \left(\frac{\Phi_{w^m}}{\Phi} - \frac{\Omega_{w^m}}{\Omega} \right) dw_t^m + \left(\frac{\Phi_g}{\Phi} - \frac{\Omega_g}{\Omega} \right) d\widehat{g}_t \\ &= o(dt) + \sigma \mathbf{w}_t' d\widehat{\mathbf{Z}}_t + \sum_{m=1}^N \left(\frac{\Phi_{w^m}}{\Phi} - \frac{\Omega_{w^m}}{\Omega} \right) w_t^m \sigma \left(d\widehat{Z}_{m,t} - \mathbf{w}_t d\widehat{\mathbf{Z}}_t \right) + \left(\frac{\Phi_g}{\Phi} - \frac{\Omega_g}{\Omega} \right) \widehat{\sigma}_t^2 \sigma^{-1} \beta' d\widehat{\mathbf{Z}}_t \end{aligned}$$

The claim follows, after noticing that the Euler equation implies the equilibrium condition

$$\mu_{M,t} = -Cov_t \left[\frac{d\pi_t}{\pi_t}, \frac{dM_t}{M_t} \right] = \sigma'_{M,t}\sigma_{\pi,t}$$

where $\sigma_{\pi,t}$ and $\sigma_{M,t}$ are defined in the claims of part (d) of Proposition A5 and A6, respectively. For an individual firm i in sector m, Ito'e Lemma on $M_t^{i,m}$ gives

$$\begin{aligned}
\frac{dM_t^{i,m}}{M_t^{i,m}} &= o(dt) + \frac{dB_t^{i,m}}{B_t^{i,m}} + \sum_{n=1}^{N} \left(\frac{\Phi_{w^n}^m}{\Phi^m} - \frac{\Omega_{w^n}}{\Omega} \right) dw_t^n + \left(\frac{\Phi_g}{\Phi} - \frac{\Omega_g}{\Omega} \right) d\widehat{g}_t \\
&= o(dt) + \sigma d\widehat{Z}_{m,t} + \sigma_1 dZ_{i,t} + \sum_{n=1}^{N} \left(\frac{\Phi_{w^n}^m}{\Phi} - \frac{\Omega_{w^n}}{\Omega} \right) w_t^n \sigma \left(d\widehat{Z}_{n,t} - \mathbf{w}_t d\widehat{\mathbf{Z}}_t \right) \\
&\quad + \left(\frac{\Phi_g}{\Phi} - \frac{\Omega_g}{\Omega} \right) \widehat{\sigma}_t^2 \sigma^{-1} \beta' d\widehat{\mathbf{Z}}_t
\end{aligned}$$

The claim follows from the Euler condition

$$\mu_{m,t} = -Cov_t \left[\frac{d\pi_t}{\pi_t}, \frac{dM_t^{i,m}}{M_t^{i,m}} \right] = \sigma'_{m,t}\sigma_{\pi,t}$$

QED (Proposition A6)

Notes on numerical solutions. For the case $N = 2$, we can compute the integrals in $\Omega(\cdot, \tau)$ and $\Phi(\cdot, \tau)$ by using standard bivariate quadrature methods. For $t < \tau$, $\Omega(\cdot, t)$ and $\Phi(\cdot, t)$ can be computed by Monte Carlo integration by jointly simulating $\widehat{g}_\tau$ and B_τ^n, for $n = 1, \ldots, N$. The latter step can be efficiently implemented using the following Lemma.

Lemma A11. Denoting by $b_t^m = \log(B_t^m)$, we have that the vector $\left(\widehat{g}_\tau, b_\tau^1, ..., b_\tau^N\right)$ is jointly normally distributed. In particular,

$$\begin{pmatrix} \widehat{g}_\tau - \widehat{g}_t \\ b_\tau^1 - b_t^1 \\ \\ b_\tau^N - b_t^N \end{pmatrix} \sim N(\mathbf{E}, \mathbf{S})$$

where the various elements of $\mathbf{E}$ and $\mathbf{S}$ are given by the following formulas:

$$\begin{aligned}
E(\widehat{g}_\tau - \widehat{g}_t) &= 0 \\
Var(\widehat{g}_\tau - \widehat{g}_t) &= \widehat{\sigma}_t^2 - \widehat{\sigma}_\tau^2 \\
E(b_\tau^m - b_t^m) &= \left(\mu - \frac{\sigma^2}{2} \right)(\tau - t) + \beta^m \widehat{g}_t (\tau - t) \\
Var(b_\tau^m - b_t^m) &= (\beta^m)^2 (\tau - t)^2 \widehat{\sigma}_t^2 + \sigma^2 (\tau - t) \\
Cov(b_\tau^m - b_t^m, b_\tau^n - b_t^n) &= \beta^m \beta^n (\tau - t)^2 \widehat{\sigma}_t^2 \\
Cov(\widehat{g}_\tau - \widehat{g}_t, b_\tau^m - b_t^m) &= \beta^m (\tau - t) \widehat{\sigma}_t^2
\end{aligned}$$

Proof of Lemma A11. The joint normality of the vector $\left(\widehat{g}_\tau - \widehat{g}_t, b^1_\tau - b^1_t, ..., b^N_\tau - b^N_t\right)$ stems from their joint dynamics:

$$d\widehat{g}_t = \widehat{\sigma}^2_t \sigma^{-1} \beta' d\widehat{\mathbf{Z}}_t \tag{B104}$$

$$db^n_t = (\mu - \frac{1}{2}\sigma^2 + \beta^n \widehat{g}_t)dt + \sigma d\widehat{Z}_{n,t} \tag{B105}$$

From Lemma A2 in the Technical Appendix of Pastor and Veronesi (2009) we have that

$$E\left(\widehat{g}_\tau - \widehat{g}_t\right) = 0 \quad \text{and} \quad V\left(\widehat{g}_\tau - \widehat{g}_t\right) = \widehat{\sigma}^2_t - \widehat{\sigma}^2_\tau$$

To see the latter, note that we can define a new Brownian motion as

$$d\overline{Z}_t = \overline{\sigma}\sigma^{-1}\beta' d\widehat{\mathbf{Z}}_t$$

where $\overline{\sigma}^2 = \sigma^2/(\beta'\beta)$. In fact, notice that $E\left[d\overline{Z}\right] = 0$ and $E\left[d\overline{Z}^2\right] = \overline{\sigma}\sigma^{-1}\beta' E\left[d\widehat{\mathbf{Z}}_t d\widehat{\mathbf{Z}}'_t\right]\beta\sigma^{-1}\overline{\sigma}dt = \overline{\sigma}^2\sigma^{-2}\beta'\beta = 1 \times dt$. Thus, we have

$$d\widehat{g}_t = \widehat{\sigma}^2_t \overline{\sigma}^{-1} d\overline{Z}_t$$

where

$$\widehat{\sigma}^2_t = \frac{1}{1/\sigma^2_g + \left(\frac{\beta'\beta}{\sigma^2}\right)t} = \frac{1}{1/\sigma^2_g + \frac{1}{\overline{\sigma}^2}t}$$

This is identical to the setting in Lemma A2 in Pastor and Veronesi (2009) and the result follows.

Turning to the capital of sector m, under the original probability measure we have

$$b^m_\tau - b_t = \left(\mu - \frac{\sigma^2}{2} + \beta^m g\right)(\tau - t) + \sigma\left(Z_{\tau,m} - Z_{t,m}\right)$$

Since $g \sim N\left(\widehat{g}_t, \widehat{\sigma}^2_t\right)$, it follows that

$$E\left(b^m_\tau - b^m_t\right) = \left(\mu - \frac{\sigma^2}{2}\right)(\tau - t) + \beta^m \widehat{g}_t (\tau - t)$$

$$Var\left(b^m_\tau - b^m_t\right) = (\beta^m)^2 (\tau - t)^2 \widehat{\sigma}^2_t + \sigma^2 (\tau - t)$$

$$Cov\left(b^m_\tau - b^m_t, b^n_\tau - b^n_t\right) = \beta^m \beta^n (\tau - t)^2 \widehat{\sigma}^2_t$$

We finally prove that

$$Cov\left(\widehat{g}_\tau - \widehat{g}_t, b^n_\tau - b^n_t\right) = \beta^n (\tau - t)\widehat{\sigma}^2_t$$

Define

$$y_t = \widehat{g}_t b^n_t$$

Ito's Lemma implies

$$\begin{aligned} dy_t &= d\widehat{g}_t b^n_t + \widehat{g}_t db^n_t + d\widehat{g}_t db^n_t \\ &= b^n_t \widehat{\sigma}^2_t \sigma^{-1}\beta' d\widehat{\mathbf{Z}}_t + \widehat{g}_t\left(\mu + \beta^n \widehat{g}_t - \frac{1}{2}\sigma^2\right)dt + \widehat{g}_t \sigma d\widehat{Z}_{n,t} + \widehat{\sigma}^2_t \beta^n dt \end{aligned}$$

Take integrals on both sides to find

$$y_\tau - y_t = \int_t^\tau b_s^n \widehat{\sigma}_s^2 \sigma^{-1} \beta' d\widehat{\mathbf{Z}}_s + \int_t^\tau \widehat{g}_s \left(\mu + \beta^n \widehat{g}_s - \frac{1}{2}\sigma^2 \right) ds + \int_t^\tau \widehat{g}_s \sigma d\widehat{Z}_{n,s} + \int_t^\tau \widehat{\sigma}_s^2 \beta^n ds$$

Taking expectations on both sides, conditional on time t information, we find

$$\begin{aligned} E_t [y_\tau - y_t] &= \int_t^\tau E_t [\widehat{g}_s] \left(\mu - \sigma^2/2 \right) + \beta^n E_t \left[\widehat{g}_s^2 \right] + \widehat{\sigma}_s^2 \beta^n ds \\ &= \int_t^\tau \widehat{g}_t \left(\mu - \sigma^2/2 \right) + \beta^n \left(\widehat{\sigma}_t^2 - \widehat{\sigma}_s^2 + \widehat{g}_t^2 \right) + \widehat{\sigma}_s^2 \beta^n ds \\ &= \widehat{g}_t \left(\mu - \sigma^2/2 \right) (\tau - t) + \int_t^\tau \beta^n \left(\widehat{\sigma}_t^2 + \widehat{g}_t^2 \right) ds \\ &= \widehat{g}_t \left(\mu - \sigma^2/2 \right) (\tau - t) + \beta^n \left(\widehat{\sigma}_t^2 + \widehat{g}_t^2 \right) (\tau - t) \end{aligned}$$

Thus, using the definition of covariance:

$$\begin{aligned} Cov \left(\widehat{g}_\tau - \widehat{g}_t, b_\tau^n - b_t^n \right) &= E_t \left[\left(\widehat{g}_\tau - \widehat{g}_t \right) \left(b_\tau^n - b_t^n \right) \right] - E_t \left[\left(\widehat{g}_\tau - \widehat{g}_t \right) \right] E_t \left[\left(b_\tau^n - b_t^n \right) \right] \\ &= E_t \left[\widehat{g}_\tau b_\tau^n - \widehat{g}_\tau b_t^n - \widehat{g}_t b_\tau^n + \widehat{g}_t b_t^n \right] \\ &= E_t \left[\widehat{g}_\tau b_\tau^n \right] - E_t \left[\widehat{g}_\tau \right] b_t^n - \widehat{g}_t E_t \left[b_\tau^n \right] + \widehat{g}_t b_t^n \\ &= \widehat{g}_t \left(\mu - \sigma^2/2 \right) (\tau - t) + \beta^n \left(\widehat{\sigma}_t^2 + \widehat{g}_t^2 \right) (\tau - t) + \widehat{g}_t b_t^n \\ &\quad - \widehat{g}_t b_t^n - \widehat{g}_t \left(b_t^n + \left(\mu - \frac{\sigma^2}{2} \right) (\tau - t) + \beta^n \widehat{g}_t (\tau - t) \right) + \widehat{g}_t b_t^n \end{aligned}$$

Simplifying common terms, we finally obtain

$$Cov \left(\widehat{g}_\tau - \widehat{g}_t, b_\tau^n - b_t^n \right) = \beta^n (\tau - t) \widehat{\sigma}_t^2$$

QED (Lemma A11).

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