Rational Panics and Stock Market Crashes*

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Abstract

This paper offers an explanation for stock market crashes which focuses on the role of rational but uninformed traders. We show that uninformed traders can precipitate a price crash because as prices decline, they surmise that informed traders received negative information, which leads them to reduce their demand for assets and drive the price of stocks even lower. The model yields several implications, such as that crashes can occur even when the fundamentals are strong, and that the magnitude of the crash depends on the fraction of uninformed investors and the amount of unsophisticated passive investing present in the market.

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Introduction

Investors are often preoccupied with the possibility that the stocks they own will arbitrarily and unexpectedly plummet in value. These fears may not be entirely unwarranted given recent historical episodes such as the October 1987 crash, when the Dow Jones index declined by 23% in a single day without any obvious corresponding changes in the underlying economic fundamentals. Not surprisingly, then, economists have devoted considerable effort to understand what forces might cause stock prices to change so dramatically in the face of seemingly small changes in fundamentals, and whether such forces are still present in markets today. At least with regard to the 1987 crash, the consensus among both policymakers and academics has converged on the culpability of then-popular hedging strategies advanced by portfolio insurance models, which dictated selling stocks when their prices began to decline. This was the theme of the Brady Report [4] commissioned to investigate the causes of the crash, and was later formalized in work by Gennotte and Leland [8]. The latter develop an equilibrium model in which crashes are impossible in the absence of portfolio insurance. They then demonstrate that a relatively small amount of portfolio insurance gives rise to a discontinuous equilibrium price function in which the equilibrium price of the risky asset jumps discretely as the underlying fundamentals change continuously.[1] Since the direct use of stocks for portfolio insurance purposes has all but disappeared, some have concluded from this analysis that markets today are less vulnerable to price crashes of similar magnitudes as in the 1987 crash.

The purpose of this paper is to take issue with this conclusion by demonstrating that the phenomenon described by Gennotte and Leland does not require the presence of exogenous portfolio insurance. We consider a market that is populated both by agents who are informed about the underlying fundamentals as well as agents who are uninformed. Thus,

[1] Gennotte and Leland further argue that the magnitude of these jumps is particularly large if traders are unaware of the presence of such strategies in the market. A similar idea is also pursued by Jacklin, Kleidon, and Pfleiderer [11].

asymmetric information plays a crucial role in our setup, in contrast with Gennotte and Leland's model where such asymmetry plays no essential role given the presence of exogenous portfolio insurance. Uninformed agents in our model can end up acting in the same way as portfolio insurers are assumed to act in Gennotte and Leland's model: they are willing to buy stocks at relatively high prices but avoid them at relatively low prices. The reason for this is that uninformed agents are aware that there are informed agents operating in the market, which gives them incentive to use prices to make inference about the underlying fundamentals. In what follows, we assume there are many informed agents active in the market, so no informed agent has incentive to hide his information. This avoids situations as in Kyle [13] or Laffont and Maskin [14] where informed traders take into account the information their actions reveal and accordingly act to conceal it. Since informed traders fail to act strategically and bid up the price of stocks when they learn fundamentals are favorable, uninformed traders correctly infer that high stock prices are more likely when fundamentals are favorable. If this effect is strong enough, uninformed agents could end up avoiding stocks altogether when stock prices are too low but not when prices are high. Thus, the demand of uninformed agents will be upward sloping. However, the demand of uninformed traders will eventually assume a downward sloping shape at high prices, both because these traders cannot afford to buy as many assets as prices continue to rise as well as because high prices eventually lead to a low rate of return. The demand of uninformed traders will therefore be upward sloping in our framework only locally.

When we add the backwards-bending demand curve of uninformed traders to the demand of informed agents, we end up with an aggregate demand curve that folds over itself, forming an inverted-S shape: at low prices, only informed and noise traders purchase assets, and their demand for the asset is downward sloping; at higher prices, the aggregate demand curve is upward sloping because of the upward sloping demand of uninformed traders; and at still higher prices, when the demand of uninformed traders turns downward sloping again, aggregate demand will be downward sloping as well. This folding implies that as we trace along values of the fundamentals, we will inevitably be forced to jump from the upper branch of the demand curve, where the demand of uninformed agents is high, to the lower branch, where the demand of the uninformed agents is low. Hence, the equilibrium price must change discontinuously at some point. This discontinuity loosely captures the notion of rational panic among uninformed traders: a small decline in price leads uninformed agents to no longer view stocks as favorable and to withdraw from the market, causing the price of the stock to fall even further. The folding property is identical to the one that Gennotte and Leland describe, except that rather than assuming exogenous hedging demand to generate this folding, our result arises endogenously under asymmetric information.

While it is well-known that demand for a good can be upward sloping in asymmetric information models – Wilson [22, 23], for example, derives an upward sloping demand in a market for goods whose quality is known only to sellers – previous work has failed to point out its role in generating stock market crashes. For one thing, simply constructing a tractable model with upward sloping demand in a market for financial assets proves to be a difficult task, in contrast with markets for goods where it is natural to assume heterogeneous taste for quality across consumers which greatly facilitates the construction of upward sloping demand schedules. In fact, Grossman [9] is able to definitively rule out upward sloping demand for financial assets under weak conditions when asset returns are normally distributed and all traders share a common utility function.

Grossman's result does not extend to models that allow for noise trading, but the strong assumptions that are often invoked to yield analytically tractable models that allow for noise do not naturally yield upward sloping demand curves either. Still, several papers have been able to generate upward-sloping demand schedules for financial assets. For example, Admati [1] uses multiple assets in a static model to generate upward sloping demand for individual assets. Wang [20] shows numerically that in a dynamic model with noise, changes in the quantity demanded by uninformed traders can be positively correlated with price changes even if there is only one risky asset, although he finds this case to be unlikely. But while both of these models give rise to upward sloping demand, they only consider linear demand schedules, so that demand for an asset never folds back as it does in our model, and the price function in both models remains continuous. By contrast, in this paper we are able to generate discontinuous price functions. This is because we adopt an alternative framework developed in Barlevy and Veronesi [3] that allows for non-linear equilibrium price functions and demand schedules. This non-linearity allows us to generate locally upward sloping demand as opposed to globally upward sloping demand, and thus to study price crashes as an equilibrium phenomenon. Although the model we use is essentially the same as the one studied in Barlevy and Veronesi [3], that paper focuses on parameter values for which upward sloping demand curves do not occur, and thus abstracts from most of the issues we consider here.

We should note that ours is not the first paper to model prices crashes in the absence of portfolio insurance. However, we believe existing models of price crashes, which focus on very different mechanisms than the one we consider, are not as convincing in explaining price crashes in equity markets. For example, Bulow and Klemperer [5] generate price crashes in a sequential auction of a finite number of commodities in which goods are sold one at a time. A crash occurs when the valuation between two agents is sufficiently far apart so that the price of the next commodity being auctioned has to fall before another agent is willing to buy it. This sequential auctioning of goods one at a time is essential for generating a crash, but it is not the way in which assets are typically traded. Madrigal and Scheinkman [16] generate a price crash that is due to strategic manipulation by a fully informed market maker who finds it optimal to coarsen the information set for potential buyers in bad states of the world, causing a discrete jump down in prices when bad outcomes occur. This would imply a stock crash is intentionally caused by a market maker, a scenario about which Madrigal and Scheinkman are themselves skeptical. A third explanation argues that small events can reveal substantial information to partially informed agents, causing traders to significantly reallocate their portfolios in response to small changes in the underlying environment, which in turn leads to large changes in stock prices. This explanation is pursued by Kraus and Smith [12], Romer [17], Caplin and Leahy [6], Lee [15], Zeira [25], and Stein and Hong [19] among others. An unattractive feature of these models is that they explain crashes as episodes in which agents learn about some underlying fundamental they were previously uncertain about and react to this information. But in practice, episodes where stock prices fall substantially appear to involve confusion and uncertainty rather than transparency and clarity. The advantage of our explanation is that it predicts crashes are driven by uninformed investors who are uncertain about the fundamentals of the asset, not by uncertain agents who suddenly become more informed about them.

The paper is organized as follows. Section 1 describes the basic layout of the model. Section 2 establishes the existence of an equilibrium in which the price is discontinuous in the underlying fundamentals, so a small change in fundamentals can be associated with an abrupt jump in the price. Section 3 discusses some implications and potential generalizations of our model. Section 4 concludes.

1. The Model

Our specification follows Barlevy and Veronesi [3], who describe an alternative set of assumptions to the traditional exponential utility and normally distributed random variables that is often employed in setting up rational expectations equilibrium models. Our only departure from the setup in that paper involves reversing a restriction on a particular parameter, which will be explained below. The assumptions we impose are quite strong, including that agents be risk-neutral, that they be precluded from short-selling, and that random variables in the model assume specific probability distributions. While these assumptions are crucial for obtaining a closed-form solution of the model, they are not all essential in generating the discontinuity that is of interest, and we comment below on why we believe the intuition is likely to generalize beyond the specific set of assumptions discussed here.

The environment we consider is static, although we discuss the possibility of extending the model into a dynamic setting further below. In our economy, agents have access to two assets. The first is a risky asset, whose price is denoted by P and which yields a random payoff of θ to its owner, where θ is binomial:

$$\theta = \begin{cases} \overline{\theta} & \text{with probability } \rho \\ \underline{\theta} & \text{with probability } 1-\rho \end{cases}$$

In Barlevy and Veronesi [3], the payoff in the bad state of the world $\underline{\theta}$ was restricted to be greater than a strictly positive lower bound.[2] Here, by contrast, we will want to allow for very low values of $\underline{\theta}$. To simplify matters, we focus on the case where $\underline{\theta} = 0$, i.e. where there exists a state of the world in which the asset is completely worthless. The analysis would continue to go through if $\underline{\theta}$ were strictly positive but sufficiently small, although the argument would be more involved and the notation far more cumbersome. The supply of this risky asset is exogenous and set equal to 1. The second asset is a safe asset whose price is normalized to 1 and which yields a fixed payoff of $1 + R$. To minimize notation, we set $R = 0$, so the safe asset can be thought of as money.

There is a continuum of agents of mass one, so that no agent can affect the price through his own trades. A fraction z of these agents are informed about the fundamentals, i.e. they know the value of θ, while the remaining $1 - z$ are uninformed and know only the distribution of θ.[3] Barlevy and Veronesi [3] derive z endogenously by assuming that agents can learn θ at a cost and show it is possible for only a fraction of the agents choose to acquire information in equilibrium. Since the source of asymmetric information is not relevant for our discussion, we suppress this in the current set-up.

Both informed and uninformed agents are initially endowed with one unit of money which

[2] In particular, they impose that $\underline{\theta} > \mu$, where μ is a distributional parameter defined below.

[3] We could equally assume informed agents observe a signal $s = \theta + \varepsilon$ with $E[\varepsilon] = 0$ rather than the actual value of the fundamentals. The demand of uninformed traders would then be given by $x^I(P, s)$ rather than $x^I(P, \theta)$, but the analysis remains unchanged.

they must allocate between the two assets. Agents are assumed to be risk-neutral. Since risk-neutral agents take infinite positions whenever the payoff to one asset exceeds the other, we need to also assume agents cannot borrow or sell assets short, i.e. the amount of assets they can purchase is bounded by the value of their original endowment.

With only informed and uninformed traders in the market, Grossman and Stiglitz [10] already demonstrated that unless we introduce some noise into the price function, the information available to informed traders will be completely revealed in prices. We therefore introduce noise traders whose trades are not driven by strategic considerations. Noise trade consists of two components, both of which serve to insure that prices are not too informative. First, we assume that noise traders follow an unsophisticated strategy of allocating a constant amount of wealth w to risky assets regardless of price. Real-world analogs of such strategies include the common practice of dollar-cost averaging or fixed contributions to retirement plans that are not conditioned on asset prices. Following this strategy, noise traders end up with $\frac{w}{P}$ shares of the asset when the price of the asset is P. We further assume that out of this amount, a random and unobserved amount x must be sold for exogenous reasons, for example to satisfy liquidity needs that require the use of money. The demand of the noise sector for risky assets can therefore be summarized by

$$x^0 = \frac{w}{P} - x \tag{1.1}$$

To insure that prices do not reveal key information on θ to uninformed traders, we need to impose two restrictions on (1.1). First, we need x to have a non-degenerate distribution so that the price is not an invertible function of θ; this way, uninformed agents cannot distinguish whether a low price is due to bad fundamentals or to low demand for assets by liquidity constrained agents. In principle, we can allow x to assume any distribution in the interval $[0, \infty)$.[4] However, to generate the type of behavior on the part of uninformed traders

[4]The restriction that $x \geq 0$ captures the fact that agents could face a pressing need to sell securities to raise cash but not a pressing need to buy them. This asymmetry is emphasized in Allen and Gorton [2].

described in the Introduction, we need a more restrictive class of distributions which satisfies certain shape restrictions that will become clear below. These restrictions are satisfied by the exponential distribution, which also turns out to be particularly convenient to analyze. Thus, we assume

$$x \sim \exp(\mu) \tag{1.2}$$

Second, we need to assume that w is sufficiently large to keep the market "liquid" and prevent prices from falling in the bad state of the world. That is, we assume

$$w > \overline{\theta} \tag{1.3}$$

This assumption implies that for any price $P \leq \overline{\theta}$, noise traders can absorb the entire net supply of assets, which recall was set fixed at 1. Without this assumption, high prices would become fully revealing, since such prices would never occur in the bad state of the world given that the demand for the risky asset is limited in that state.

To recap, total demand for risky assets is equal to the sum of the demand of the risk-neutral agents (informed and uninformed) and noise trade x^0. The total supply of risky assets is equal to 1. Agents trade given their information, and the market clears so that supply equals demand. A rational expectations equilibrium in this market is a price function and demand correspondences $\left(P,\ x^I,\ x^U\right)$ that satisfy the usual optimality and market clearing conditions. Specifically, $P(x,\theta)$ denotes the price of the risky asset for every state of the world given by the pair (x,θ); $x^I\left(\theta,P\right)$ denotes the demand of informed traders for the risky asset when they know the value of the asset is θ and the price of the asset is P; and $x^U\left(P;P(\cdot,\cdot)\right)$ is the demand of uninformed traders for the risky asset when the price of the asset is P, conditional on knowing that the equilibrium price function is $P(x,\theta)$. Formally:

Definition: A rational expectations equilibrium for the economy is a set of functions $\left(P\left(x,\theta\right), x^I\left(\theta,P\right), x^U\left(P;P\left(\cdot,\cdot\right)\right)\right)$ such that

1. **Utility Maximization**: x^I and x^U solve the maximization problem of the agents conditional upon their information:

$$\max_{x^i} E_i\left[\theta\right] x^i + \left(1 - Px^i\right)$$

2. **Market Clearing** For all pairs (x, θ), the price $P(x, \theta)$ equates supply and demand, i.e.

$$x^0 \; + \; \int_0^z x_j^I\left(\theta, P(x,\theta)\right) dj \; + \; \int_z^1 x_j^U\left(P(x,\theta); P(\cdot,\cdot)\right) dj = 1$$

where $j \in [0, 1]$ indexes individual agents.

2. Stock Market Crashes

Following Gennotte and Leland [8], we define a price crash as a scenario in which the equilibrium price function $P(\cdot,\cdot)$ is discontinuous in any of its continuous arguments, which in this case involves only the noise variable x. This definition captures the essential element of a crash that a small change in the underlying fundamentals is associated with a disproportionately large change in the price of the asset. Of course, this definition captures crashes in only a very rough sense, since our model is inherently static, and it is difficult to extrapolate from this model an explanation for why the price of a given asset could change dramatically *over time* despite relatively small changes in the fundamentals. We will comment on dynamic interpretations and extensions of our results further below, but for now we simply adopt this definition since it is the most appropriate for the static environment the model describes.[5]

To guide the reader, it will help to lay out the organization of this section in advance. We begin by conjecturing what the demand for assets among uninformed agents might look

[5] Note that this definition can be equally interpreted as a crash or as an upward jump in the price. Several authors have argued that a reasonable theory of crashes should account for the apparent asymmetry in stock prices changes, which this approach cannot do. For an opposing view on whether price changes should be viewed as asymmetric, see the discussion in Gennotte and Leland, especially page 1013.

like in equilibrium. We then derive the set of prices $\widetilde{P}(x,\theta)$ that would clear the market under this demand schedule. We show that for the demand schedule we start with, the set $\widetilde{P}(x,\theta)$ is folded over itself, which implies the price function in any rational expectations equilibrium associated with the demand schedule we originally conjectured will necessarily be discontinuous. Finally, we show that there is a price function from the set $\widetilde{P}(x,\theta)$ for which the demand schedule we initially conjectured is optimal. This establishes the existence of a rational expectations equilibrium in which the equilibrium price function is discontinuous in x. While this proof by construction is useful for understanding how the price schedule and the demand of uninformed traders are related in equilibrium, it only characterizes one equilibrium out of many which can exist in this economy. At the end of this section, we characterize the set of all rational expectations equilibria and argue that there exists no equilibrium in which the price function is continuous in x. Hence, equilibrium crashes are unavoidable in our environment, although their size and nature are not uniquely determined.

2.1. Demand Schedules

We begin by considering what the demand for the risky asset might look like in equilibrium. Since traders are risk neutral, they allocate all of their endowment to the asset that yields the higher expected payoff. For informed traders who know the value of θ, the optimal demand schedule in any rational expectations equilibrium is uniquely given by

$$x^{I}(\theta,P)\in\begin{cases}0 & \text{if}\quad P>\theta\\ \left[0,\dfrac{1}{P}\right] & \text{if}\quad P=\theta\\ \dfrac{1}{P} & \text{if}\quad P<\theta\end{cases}\tag{2.1}$$

Uninformed traders will similarly allocate all of their endowment to the asset that yields the higher expected payoff, i.e.

$$x^{U}(P;P(\cdot,\cdot))\in\begin{cases}0 & \text{if}\quad P>E\left[\theta\mid P(x,\theta)=P\right]\\ \left[0,\dfrac{1}{P}\right] & \text{if}\quad P=E\left[\theta\mid P(x,\theta)=P\right]\\ \dfrac{1}{P} & \text{if}\quad P<E\left[\theta\mid P(x,\theta)=P\right]\end{cases}\tag{2.2}$$

Since the equilibrium demand schedule of the uninformed traders depends on the entire equilibrium price function $P(x,\theta)$, we have to simultaneously derive the optimal demand schedule for uninformed traders together with the price function $P(\cdot,\cdot)$. We first attack this problem by conjecturing what the demand schedule for uninformed traders might look like. We then proceed to confirm that there exists a market clearing price function $P(x,\theta)$ that equates this demand with supply. Lastly, we show that given this price function, it will be optimal for uninformed traders to submit the demand schedule we conjecture.

To generate crashes, we need the demand curve of uninformed traders be locally upward sloping in prices. The simplest such demand schedule given risk-neutral preferences is if uninformed traders rely on two cutoffs P' and P'' where $P'' > P' > 0$, such that they allocate all of their wealth to money for prices below P' and above P'', and to risky assets for prices in the interval (P', P''). Formally, we conjecture the demand schedule of the uninformed is given by

$$x^U(P) \in \begin{cases} 0 & \text{if} \quad P < P' \\ \left[0, \frac{1}{P}\right] & \text{if} \quad P = P' \\ \frac{1}{P} & \text{if} \quad P \in (P', P'') \\ \left[0, \frac{1}{P}\right] & \text{if} \quad P = P'' \\ 0 & \text{if} \quad P > P'' \end{cases} \tag{2.3}$$

This demand schedule is illustrated graphically on the left side of Figure 1. The cutoff P' generates a locally upward sloping demand curve, since uninformed traders demand a positive amount of the risky asset at prices just above P' but not at prices just below P'. We further conjecture there is a second cutoff P'', so that uninformed traders are once again unwilling to purchase the asset when its price is very high. This second cutoff is inevitable: any upward sloping demand curve among uninformed traders would have to bend backwards eventually if θ has bounded support, since demand would fall to zero when the price of the asset exceeds the maximum realization of θ. This distinguishes our model from some of the previous work we cite above where θ is unbounded, so demand for assets could be everywhere upward sloping and still remain optimal. The fact that the demand curve of the uninformed is both upwards sloping and downward sloping will prove to be important in what follows.

Thus far, we only guess that (2.3) forms an optimal demand schedule, although we have not yet demonstrated this will be the case. However, intuitively, it is not unreasonable to conjecture that the information conveyed by low prices might be sufficient to deter uninformed agents from buying the asset, since x would have to be unusually large to offset the demand of informed traders when θ is high to keep the price so low, making that scenario rather unlikely. We will eventually verify this by demonstrating that there does exist an equilibrium price function $P(x,\theta)$ for which the above demand curve is optimal.

2.2. Market-Clearing Prices

Having guessed what the demand schedule for uninformed traders might look like in equilibrium, our next step in solving for an equilibrium is to figure out which prices would clear the market given this demand. Let $\widetilde{P}(x,\theta)$ denote the set of all prices that clear the market when the fundamentals are equal to x and θ, given the demand schedule in (2.3), i.e.

$$\widetilde{P}(x,\theta) = \left\{ P \;\middle|\; \frac{w}{P} - x + \int_0^z x_j^I(P,\theta)\,dj + \int_z^1 x_j^U(P)\,dj = 1 \right\}$$

The set $\widetilde{P}(x,\theta)$ can be constructed in steps by conditioning on different possible prices P. For example, consider prices $P > \overline{\theta}$. At such prices, neither informed nor uninformed traders will purchase the risky asset, and so market clearing would imply $\frac{w}{P} - x = 1$. In this case, the unique price that clears the market is given by $P = \frac{w}{x+1}$. Hence, for all x such that $\frac{w}{x+1} > \overline{\theta}$, $\frac{w}{x+1} \in \widetilde{P}(x,\theta)$ for either value of θ. Proceeding similarly for the cases where the price is equal to the cutoffs $\overline{\theta}$, P', and P'', as well as values in the intervals defined by these cutoffs, we can construct the entire set $\widetilde{P}(x,\theta)$. This set is illustrated graphically in the right hand side of Figure 1, and is characterized below in (2.4) and (2.5). $\widetilde{P}(x,\theta)$ consists of several piecewise hyperbolic functions defined in the figure, along with flat segments at cutoff prices where some agents are indifferent as to how they allocate their wealth.

As can be seen in Figure 1, there is a range of x for which the set $\widetilde{P}(x,\theta)$ contains more than one market clearing price. The source of multiplicity can be understood as follows. Suppose we start with an equilibrium in which the price is just below P' and uninformed agents are not purchasing the asset. If we now force all uninformed agents to purchase the asset, they will drive the price to just above P', which in turn validates their decision to allocate their wealth to risky assets and yields another equilibrium. Formally, let us rewrite the market clearing condition so that demand is given by the sum of unsophisticated demand and the demand of risk-neutral agents, i.e.

$$D(P) = \frac{w}{P} + \int_0^z x_j^I(P,\theta)\,dj + \int_z^1 x_j^U(P)\,dj$$

while supply is given by a constant, $1+x$, i.e. the endowment of assets plus the supply of liquidity constrained traders. The aggregate demand schedule $D(P)$ is non-monotonic and continuous in P at P'. The restriction that $w > \overline{P}$ implies that $D(P) > 1$ for all $P < P''$. It follows that there must exist an $x > 0$ for which there are at least two values of P such that $D(P) = 1 + x$. Note that all that is required is that $D(P)$ be continuous and locally non-monotonic; the fact that the demand of uninformed agents exhibits a discrete jump at P' is in no way essential for this argument. This suggests that the bang-bang nature of demand under risk-neutrality is probably not crucial for our analysis, and risk-averse preferences could in principle generate a similar type of multiplicity. However, abandoning risk-neutrality would render the problem of solving for equilibrium mathematically intractable, which is why we are forced to impose it.

The fact that there are multiple market clearing prices for a given realization of (x,θ) could directly account for price crashes, since traders could switch from one equilibrium price to another without any change in the fundamentals. However, since our definition for a rational expectations equilibrium assumes a price *function*, we implicitly assign a unique price for each pair (x,θ) and thus rule out this explanation. While multiple equilibria could in principle account for observed price crashes, we are instead interested in a different question: even if we required a single price for each realization of the fundamentals so that prices could not arbitrarily change for a given set of fundamentals, what type of price functions could arise in equilibrium in our economy, and would such price functions necessarily be discontinuous in the underlying fundamentals? Here, it proves to be important that $D(P)$ is only locally upward sloping. At very low prices, only informed and noise traders ever demand the risky asset, and their demand schedules are downward sloping. At very high prices, the demand of all three agents, including the uninformed, will all be downward sloping. Thus, the graph of $\widetilde{P}(x, \theta)$ will fold over itself in an interval of x, forming an inverted-S pattern. This structure is commonly referred to as a fold catastrophe in the mathematical literature on discontinuities.[6] It implies that any selection that assigns a unique price for each x would necessarily generate a discontinuous price function, i.e. it is not possible to choose a function $P(x, \theta)$ from the set $\widetilde{P}(x, \theta)$ that is continuous in x. To obtain an intuition for this result, recall that the upper branch corresponds to the equilibrium in which uninformed traders are willing to purchase the asset (at least when prices are not too high), while the lower branch corresponds to the equilibrium in which uninformed traders avoid the asset. Starting at the top branch, as we increase x, we increase the effective supply of risky assets and depress the price of the asset. At some point, prices are depressed enough so that even if uninformed agents allocate all of their wealth to risky assets, the price cannot be above $\underline{P}$, implying there can no longer be a market clearing price at which uninformed traders still desire the asset. This explains why we refer to this phenomenon as a rational panic: a small decline in price due to increased supply from noise traders leads uninformed traders to reduce their demand for stocks given their perception of the equilibrium price function, driving the price down further still.

2.3. Existence of Equilibrium

Our analysis so far has established that any price function which satisfies market clearing given the demand curve in (2.3) must be discontinuous. But we have yet to show that

[6] See, for example, Rosser [18].

there actually exists a discontinuous price function that we can choose from the set $\widetilde{P}(x,\theta)$ for which the demand schedule we originally conjectured is indeed optimal. The next proposition establishes that for sufficiently large values of ρ, there exists at least one selection $P(x,\theta)$ from the set $\widetilde{P}(x,\theta)$ that supports the demand schedule (2.3) as optimal. First, though, we establish the following Lemma:

Lemma 1: There exists a $\rho^* \in (0,1)$ such that for all $\rho > \rho^*$, there exist two unique levels $\underline{P}$ and $\overline{P}$ such that for all $P > 0$,

$$\frac{\rho\overline{\theta}}{\rho + (1-\rho)\exp(\mu z/P)} > P \iff P \in \left(\underline{P}, \overline{P}\right)$$

Proof: Let $y = \dfrac{1}{P}$ and define $h(y) = \rho\overline{\theta}y - [\rho + (1-\rho)\exp(\mu z y)]$. It follows that $\frac{\rho\overline{\theta}}{\rho+(1-\rho)\exp(\mu z/P)} > P$ if and only if $h(y) > 0$. It is clear that $h(y)$ is strictly concave, so that it can equal 0 at most twice. For any $y \leq 1/\overline{\theta}$, we have $h(y) < -(1-\rho)\exp(\mu z y) < 0$. In the opposite direction, $\lim_{y\to\infty} h(y) = -\infty$. It remains to show that there exists a $y \in \left(1/\overline{\theta}, \infty\right)$ such that $h(y) > 0$. But for any $y > 1/\overline{\theta}$, $\lim_{\rho\to 1} h(y) = \overline{\theta}y - 1 > 0$, so that there exists a $\rho^* \in (0,1)$ such that for all $\rho > \rho^*$, there is a y such that $h(y) = 0$. It follows that for a given $\rho > \rho^*$, there are exactly two values of $y \in \left(1/\overline{\theta}, \infty\right)$ for which $h(y) = 0$. Given μ, z, and ρ, we can define $\underline{P}$ and $\overline{P}$ as the inverse of these roots for the function $h(y)$. ■

In the next Proposition, we use the two roots $\underline{P}$ and $\overline{P}$ from Lemma 1 to form the cutoffs in the demand of the uninformed. Specifically, we show that if we set $P' = \underline{P}$ and $P'' = \overline{P}$, we can construct a price function $P(x,\theta)$ from the set of market clearing prices $\widetilde{P}(x,\theta)$ for which the demand schedule of uninformed (2.3) we originally conjectured is indeed optimal.

Proposition 1: For $\rho > \rho^*$ from Lemma 1, there exists a price function $P(x,\theta) \in \widetilde{P}(x,\theta)$ for which the demand schedule (2.3) is optimal. Hence, for all $\rho > \rho^*$, there exists a rational expectations equilibrium in which the price function $P(x,\theta)$ is discontinuous in x.

Proof: Set $P' = \underline{P}$ and $P'' = \overline{P}$ from Lemma 1. The piecewise hyperbolic correspondence $\widetilde{P}(x,\theta)$ described in Figure 1 is then given by

$$\widetilde{P}(x,\underline{\theta}) = \begin{cases} \frac{w}{x+1} & \text{for} \quad x+1 < \frac{w}{\overline{P}} \\ \overline{P} & \text{for} \quad x+1 \in \left[\frac{w}{\overline{P}}, \frac{w+1-z}{\overline{P}}\right] \\ \frac{w+1-z}{x+1} & \text{for} \quad x+1 \in \left(\frac{w+1-z}{\overline{P}}, \frac{w+1-z}{\underline{P}}\right) \\ \underline{P} & \text{for} \quad x+1 \in \left[\frac{w}{\underline{P}}, \frac{w+1-z}{\underline{P}}\right] \\ \frac{w}{x+1} & \text{for} \quad x+1 \geq \frac{w}{\underline{P}} \end{cases} \tag{2.4}$$

$$\widetilde{P}(x,\overline{\theta}) = \begin{cases} \frac{w}{x+1} & \text{for} \quad x+1 < \frac{w}{\overline{\theta}} \\ \overline{\theta} & \text{for} \quad x+1 \in \left[\frac{w}{\overline{\theta}}, \frac{w+z}{\overline{\theta}}\right] \\ \frac{w+z}{x+1} & \text{for} \quad x+1 \in \left(\frac{w+z}{\overline{\theta}}, \frac{w+z}{\overline{P}}\right) \\ \overline{P} & \text{for} \quad x+1 \in \left[\frac{w+z}{\overline{P}}, \frac{w+1}{\overline{P}}\right] \\ \frac{w+1}{x+1} & \text{for} \quad x+1 \in \left(\frac{w+1}{\overline{P}}, \frac{w+1}{\underline{P}}\right) \\ \underline{P} & \text{for} \quad x+1 \in \left[\frac{w+z}{\underline{P}}, \frac{w+1}{\underline{P}}\right] \\ \frac{w+z}{x+1} & \text{for} \quad x+1 \geq \frac{w+z}{\underline{P}} \end{cases} \tag{2.5}$$

Since $\underline{P} < \overline{P} < \overline{\theta}$, $z \in (0,1)$, and $w > \overline{\theta} > \overline{P}$, all of these intervals are non-degenerate. There is no unique way to choose $P(x,\theta)$ from the above set. We therefore choose one particular $P(x,\theta)$, which is depicted graphically in Figure 2, although it is certainly not the only selection that forms an equilibrium together with the demand schedule in (2.3). We begin with $\theta = \overline{\theta}$, and assign $P\left(x,\overline{\theta}\right)$ continuously along the upper branch as far as we can, i.e. for all $x < \frac{w+1}{\underline{P}} - 1$. At $x = \frac{w+1}{\underline{P}} - 1$, we shift to the lower branch, and assign prices continuously from there on. This implies that $x = \frac{w+1}{\underline{P}} - 1$ is the only point of discontinuity for $P\left(\cdot,\overline{\theta}\right)$. In particular,

$$\begin{aligned} \lim_{x \to \left(\frac{w+1}{\underline{P}}-1\right)^-} P\left(x,\overline{\theta}\right) &= \underline{P} \\ \lim_{x \to \left(\frac{w+1}{\underline{P}}-1\right)^+} P\left(x,\overline{\theta}\right) &= \frac{w+z}{w+1}\underline{P} \equiv P^* \end{aligned}$$

The point $P^* < \underline{P}$ is indicated in Figure 2. For $\theta = \underline{\theta}$, we construct $P(x,\underline{\theta})$ by moving in the opposite direction; we assign $P(x,\underline{\theta})$ continuously along the lower branch for all

$x \geq \frac{w}{P^*} - 1$, and for $x < \frac{w}{P^*} - 1$, we assign $P(x, \underline{\theta})$ along the upper branch. Once again, there is a single discontinuity, this time at the point $x = \frac{w}{P^*} - 1$. In particular,

$$\begin{aligned} \lim_{x \to \left(\frac{w+1}{P^*} - 1\right)^-} P(x, \theta) &= \frac{w+1-z}{w} P^* \\ \lim_{x \to \left(\frac{w+1}{P^*} - 1\right)^+} P(x, \theta) &= P^* \end{aligned}$$

Note that for $z \in (0,1)$, we have

$$\begin{aligned} \frac{w+1-z}{w} P^* &= \frac{w+1-z}{w} \frac{w+z}{w+1} \underline{P} \\ &= \left[1 + \frac{z - z^2}{w + w^2}\right] \underline{P} > \underline{P} \end{aligned}$$

It follows that $\{P(x, \underline{\theta}) \mid x \geq 0\} \subset \left\{P\left(x, \overline{\theta}\right) \mid x \geq 0\right\}$, i.e. the range of prices that arise in the bad state of the world is a strict subset of the range of prices that arise in the good state of the world. Prices in the non-empty interval $\left(\underline{P}, \frac{w+1-z}{w} P^*\right]$ only arise in the good state, and are therefore fully revealing. This can also be seen in the left hand side of Figure 2.

Using Bayes' rule, we can compute the conditional expectation $E(\theta \mid P(\cdot,\cdot) = P)$ for the price function $P(\cdot,\cdot)$ we construct:

$$E(\theta \mid P(\cdot,\cdot) = P) = \begin{cases} \rho\overline{\theta} & \text{if} \quad P > \overline{\theta} \\ \overline{\theta} & \text{if} \quad P = \overline{\theta} \\ \dfrac{\rho\overline{\theta}}{\rho + (1-\rho)\phi(P)} & \text{if} \quad P \in \left(\dfrac{w+1-z}{w} P^*, \overline{\theta}\right) \\ \overline{\theta} & \text{if} \quad P \in \left(\underline{P}, \dfrac{w+1-z}{w} P^*\right] \\ \dfrac{\rho\overline{\theta}}{\rho + (1-\rho)\phi(P)} & \text{if} \quad P \leq P^* \end{cases} \tag{2.6}$$

where

$$\phi(P) = \frac{\text{Prob}(P(\cdot,\cdot) = P \mid \theta = \underline{\theta})}{\text{Prob}\left(P(\cdot,\cdot) = P \mid \theta = \overline{\theta}\right)}$$

is the likelihood ratio of observing the price P under the different values of θ. This just corresponds to a likelihood ratio for different values of x, namely

$$\phi(P) = \frac{f(x(P, \underline{\theta}))}{f\left(x(P, \underline{\theta}) + \frac{z}{P}\right)} \tag{2.7}$$

where $x(P, \underline{\theta})$ represents the value of x for which $P(x, \underline{\theta}) = P$. For the exponential distribution, the likelihood ratio is conveniently independent of the value of $x(\cdot, \cdot)$:

$$\phi(P) = \exp\left(\frac{\mu z}{P}\right)$$

To assess whether (2.3) is optimal, we need to compare $E(\theta \mid P(\cdot, \cdot) = P)$ with P. If $P \geq \overline{\theta}$ or $P \in \left(\underline{P}, \frac{w+1-z}{w} P^*\right]$, the demand schedule is trivially optimal. The remaining cases hinge on how $\frac{\rho \overline{\theta}}{\rho + (1-\rho)\phi(P)}$ compares with P. However, for the exponential distribution case, where $\phi(P) = \exp\left[\frac{\mu z}{P}\right]$, Lemma 1 establishes that the former is greater than P if and only if $P \in \left(\underline{P}, \overline{P}\right)$. Since $P(x, \theta) \neq \underline{P}$ for all (x, θ) by construction (note the open circle in the right hand side of Figure 2 at $P = \underline{P}$) , any demand is optimal at that price. For all remaining prices, it is easy to verify that (2.3) is indeed optimal given the expectation and the definition of $\underline{P}$ and $\overline{P}$. ■

To summarize, if the probability that $\theta = \overline{\theta}$ is sufficiently large, uninformed traders will optimally avoid assets at both high and low prices but purchase them at intermediate prices. The fact that assets are desirable at intermediate prices follows from the fact that if we pick any price $P \in (0, \overline{\theta})$ and take the limit as ρ approaches 1, the conditional expectation $E(\theta \mid P(\cdot, \cdot) = P) \to \overline{\theta} > P$, insuring the risky asset will be attractive at some positive price. However, agents will not necessarily agree to purchase the asset at any price below $\overline{\theta}$. If the conditional expectation $E[\theta \mid P(\cdot, \cdot) = P]$ declines rapidly with the price of the asset P, the return on the risky asset could fall below the return on money at low prices. For example, suppose the distribution of the noise term x has full support over $[0, \infty)$, i.e. $f(x) > 0$ for all $x \geq 0$. This implies that the likelihood ratio $\phi(P)$ for $\theta = \underline{\theta}$ shoots off to infinity as the price falls towards zero, which in turn implies $\lim_{P \to 0} E[\theta \mid P(\cdot, \cdot) = P] = 0$. Hence, when the asset is free, the agent does not expect it to pay off, and there is no particular incentive to hold it. As long as the distribution of x further implies that

$$\lim_{P \to 0} \frac{\partial}{\partial P} E[\theta \mid P(\cdot, \cdot) = P] < 1 \tag{2.8}$$

we can ensure there is no incentive to hold the asset even for prices just above zero, since money offers a higher expected return. Condition (2.8) amounts to an implicit restriction on the tail distribution of the noise term x. In particular, the right tail of the distribution must converge rapidly enough to zero that the likelihood ratio $\phi(P)$ explodes quickly as $P \to 0$, implying very low values for the conditional expectation of θ at small but positive prices. This is true of the exponential distribution, where the tail decays at an exponential rate, but the same would be true for any other distribution where the tail of the distribution converges fast enough towards zero, e.g. a truncated normal. The fact that upward sloping demand depends on conditions that govern the distribution of fundamentals mirrors corresponding results in the asymmetric information literature, such as Wilson [22, 23], who also finds that the existence of an upward sloping demand depends on the exogenous distribution of market fundamentals.

2.4. Characterizing the Set of Equilibria

Our discussion above offers a simple algorithm to solve for an equilibrium: we conjecture a demand schedule for uninformed traders and then confirm that the set of prices which clear the market for this demand imply that our conjecture is indeed an optimal demand schedule. The limitation of this approach is that it only allows us to confirm whether a particular configuration is an equilibrium rather than to solve for all possible equilibria of the model. Clearly, the equilibrium in Proposition 1 is not unique; there is a continuum of selections from $\widetilde{P}(x, \theta)$ that are consistent with the optimality of our conjectured demand curve. But there may be other equilibria associated with different demand schedules, which may not be associated with discontinuous equilibrium price functions. We now show that our assumptions allow us to characterize the entire set of equilibrium prices. While neither the price function $P(x, \theta)$ nor the demand schedule $x^U(P)$ are uniquely determined in equilibrium, we show that in *any* equilibrium the demand schedule must be locally upward sloping, and the equilibrium price function will exhibit at least one discontinuity. Thus, price crashes are an inherent property of any equilibrium in our model.

The reason we can still characterize the set of equilibria quite easily in our model is that we assume the noise term x is distributed as an exponential. Consequently, the likelihood ratio $\phi(P)$ described in the proof of Proposition 1 depends only on the price and not on $x(P,\underline{\theta})$, i.e. $\phi(x(P,\underline{\theta}), P) = \phi(P)$. This allows us to deduce the expected return on the risky asset anticipated by uninformed traders without explicitly solving their entire demand schedule in equilibrium; for alternative distributions of x, we would have to compute it in order to solve $x(P,\underline{\theta})$. We begin with a lemma that establishes uninformed traders can draw only one of three inferences on the expected value of θ when observing a price in the range of the equilibrium price function.

Lemma 2: Consider a price $P < \overline{\theta}$ such that $P \in \text{range}\{P(\cdot,\cdot)\}$. In any rational expectations equilibrium,

1. If $P \in \text{range}\{P(\cdot,\underline{\theta})\} \cap \text{range}\{P(\cdot,\overline{\theta})\}$, then $E[\theta \mid P(x,\theta) = P] = \dfrac{\rho\overline{\theta}}{\rho + (1-\rho)\exp(\mu z/P)}$
2. If $P \in \text{range}\{P(\cdot,\underline{\theta})\}$ and $P \notin \text{range}\{P(\cdot,\overline{\theta})\}$, then $E[\theta \mid P(x,\theta) = P] = 0$
3. If $P \notin \text{range}\{P(\cdot,\underline{\theta})\}$ and $P \in \text{range}\{P(\cdot,\overline{\theta})\}$, then $E[\theta \mid P(x,\theta) = P] = \overline{\theta}$

Proof: Parts (2) and (3) follow trivially. For part (1), define $m^U(P(x,\theta))$ as the amount of wealth uninformed traders spend in equilibrium on risky assets when the realized price is equal to $P(x,\theta)$. Market clearing establishes that the price function will be given by

$$P(x,\theta) = \begin{cases} \dfrac{w+z+m^U}{x+1} & \text{if } \theta = \overline{\theta} \\[2ex] \dfrac{w+m^U}{x+1} & \text{if } \theta = \underline{\theta} \end{cases}$$

Define the set $X + y$ for a set X and a point y as $\{x + y \mid x \in X\}$. It follows that

$$\{x \mid P(x,\underline{\theta}) = P\} = \{x \mid P(x,\overline{\theta}) = P\} + \frac{z}{P} \tag{2.9}$$

This is because for any x for which $P(x, \underline{\theta}) = P$ there must be a distinct x' such that $P(x', \overline{\theta}) = P$. Using Bayes' rule, the conditional probably of the good state of the world is given by

$$\text{Prob}\left(\theta = \overline{\theta} \mid P(\cdot,\cdot) = P\right) = \frac{\rho \int_{\left\{x \mid P\left(x,\overline{\theta}\right)=P\right\}} f(x)\, dx}{\rho \int_{\left\{x \mid P\left(x,\overline{\theta}\right)=P\right\}} f(x)\, dx + (1-\rho) \int_{\left\{x \mid P\left(x,\underline{\theta}\right)=P\right\}} f(x)\, dx}$$

Using (2.9) and the fact that $f(x) = \mu \exp(-\mu x)$, we find

$$\text{Prob}\left(\theta = \overline{\theta} \mid P(\cdot,\cdot) = P\right) = \frac{\rho}{\rho + (1-\rho)\exp\left(\frac{\mu z}{P}\right)}$$

so that for any $P < \overline{\theta}$, $E\left(\theta = \overline{\theta} \mid P(\cdot,\cdot) = P\right) = \dfrac{\rho\overline{\theta}}{\rho + (1-\rho)\exp(\mu z/P)}$ as claimed. ■

Recall from Lemma 1 that for $\rho > \rho^*$, there exist two unique roots $\underline{P}$ and $\overline{P}$ such that $\frac{\rho\overline{\theta}}{\rho+(1-\rho)\exp(\mu z/P)} > P$ if and only if $P \in \left(\underline{P}, \overline{P}\right)$. We use this fact to argue that all equilibrium price functions represent a selection from a correspondence $\widehat{P}(x,\theta)$ that can be constructed using $\underline{P}$ and $\overline{P}$.

Proposition 2: Suppose $\rho > \rho^*$, as defined in Lemma 1. Then there exists a unique correspondence $\widehat{P}(x,\theta)$ such that any rational expectations equilibrium price $P(x,\theta)$ is contained in $\widehat{P}(x,\theta)$, and for any element of $\widehat{P}(x,\theta)$ there exists an equilibrium price $P(x,\theta)$ that contains that element.

Proof: The proof is again by construction. In particular, we use the fact that the market clearing condition implies

$$P(x,\theta) = \frac{w + m^I(\theta, P) + m^U(P)}{x+1}$$

where $m^I(\theta, P)$ denotes the total wealth spent by informed agents on the risky asset, and $m^U(P)$ denotes the total wealth spent by uninformed agents. The demand of informed agents can be derived from (2.1). The key is the equilibrium restrictions on $m^U(P)$.

First, we argue that for any price $P \in \left(\overline{P}, \overline{\theta}\right)$, the amount of wealth $m^U(P)$ allocated to risky assets by uninformed traders must equal zero. The market clearing condition implies

$$P(x, \underline{\theta}) = \frac{w + m^U(P(x, \underline{\theta}))}{x+1} \geq \frac{w}{x+1}$$

Hence, for $x < \frac{w}{\overline{P}} - 1$, any market clearing price $P(x, \underline{\theta}) > \overline{P}$. Let P' denote the value of $P(x, \underline{\theta})$. There are two possible cases. First, $P' \in \text{range}\left\{P\left(\cdot, \overline{\theta}\right)\right\}$, in which case

$$E\left[\theta \mid P(x, \theta) = P'\right] = \frac{\rho\overline{\theta}}{\rho + (1-\rho)\phi(P')} < P'$$

since $P' > \overline{P}$. In this case, it would be optimal for an uninformed agent to purchase only money. Alternatively, $P \notin \text{range}\left\{P\left(\cdot, \overline{\theta}\right)\right\}$, in which case

$$E\left[\theta \mid P(x, \theta) = P'\right] = 0 < P'$$

and again it is optimal to allocate all wealth to money. Hence, for any $x < \frac{w}{\overline{P}} - 1$, it follows that $m^U(P(x, \underline{\theta})) = 0$ and $P(x, \underline{\theta}) = \frac{w}{x+1}$. This uniquely establishes that $\widehat{P}(x, \underline{\theta}) = \frac{w}{x+1}$ for $x+1 \in \left(\frac{w}{\overline{\theta}}, \frac{w}{\overline{P}}\right)$. A similar argument uniquely pins down $\widehat{P}(x, \underline{\theta}) = \frac{w}{x+1}$ for $x+1 \leq \frac{w}{\overline{\theta}}$. Given the restriction on $m^U(P)$ for all $P \in \left(\overline{P}, \overline{\theta}\right)$, we can uniquely pin down $\widehat{P}\left(x, \overline{\theta}\right) = \frac{w+z}{x+1}$ for $x+1 \in \left(\frac{w+z}{\overline{\theta}}, \frac{w+z}{\overline{P}}\right)$, as well as that $\widehat{P}\left(x, \overline{\theta}\right) = \overline{\theta}$ for $x+1 \in \left(\frac{w}{\overline{\theta}}, \frac{w+z}{\overline{\theta}}\right)$.

Next, we argue that for any price $P < \frac{w\underline{P}}{w+1-z}$, $m^U(P) = 0$. Once again, from the market clearing condition, we have

$$P(x, \underline{\theta}) = \frac{w + m^U(P(x, \underline{\theta}))}{x+1} \leq \frac{w+1-z}{x+1}$$

since $m^U(\cdot)$ is bounded above by $1-z$. Hence, for any $x > \frac{w+1}{\underline{P}} - 1$, $P(x, \underline{\theta}) < \underline{P}$. By a similar argument as above, we can establish that for such values of x, $m^U(P(x, \underline{\theta})) = 0$ and $P(x, \underline{\theta}) = \frac{w}{x+1}$. This uniquely establishes $\widehat{P}(x, \underline{\theta}) = \frac{w}{x+1}$ for $x+1 > \frac{w+1-z}{\underline{P}}$. Given the restriction on $m^U(P)$ for all $P < \frac{w\underline{P}}{w+1-z}$, we can uniquely pin down $\widehat{P}\left(x, \overline{\theta}\right) = \frac{w+z}{x+1}$ for $x+1 > \frac{(w+1-z)(w+1)}{w\underline{P}}$, since for any $m^U(P) \in [0, 1-z]$, the price function $P\left(x, \overline{\theta}\right) < \frac{w\underline{P}}{w+1-z}$.

Lastly, we argue that for any price $P \in \left(\frac{w+1}{w+z}\underline{P}, \overline{P}\right)$, the amount of wealth $m^U(P)$ allocated to risky assets must equal $1-z$. Again, market clearing requires

$$P\left(x,\overline{\theta}\right) = \frac{w+z+m^U\left(P\left(x,\overline{\theta}\right)\right)}{x+1} \in \left[\frac{w+z}{x+1}, \frac{w+1}{x+1}\right]$$

Hence, for all x such that $x+1 \in \left(\frac{w+1}{\overline{P}}, \frac{w+z}{\underline{P}}\right)$, $P\left(x,\overline{\theta}\right) \in \left(\underline{P}, \overline{P}\right)$. It follows that $m^U\left(P\left(x,\overline{\theta}\right)\right) = 1-z$, which pins down $\widehat{P}\left(x,\overline{\theta}\right) = \frac{w+1}{x+1}$. Just as before, we can also uniquely pin down $\widehat{P}\left(x,\underline{\theta}\right) = \frac{w+1-z}{x+1}$ for $x+1 \in \left(\frac{w+1-z}{\overline{P}}, \frac{(w+z)w}{(w+1)\underline{P}}\right)$. It is then straightforward to show that $\widehat{P}\left(x,\overline{\theta}\right) = \overline{P}$ for $\left(\frac{w+z}{\overline{P}}, \frac{w+1}{\overline{P}}\right)$ and $\widehat{P}\left(x,\underline{\theta}\right) = \overline{P}$ for $\left(\frac{w}{\overline{P}}, \frac{w+1-z}{\overline{P}}\right)$.

For the remaining values of x, it is impossible to pin down the equilibrium price $P(x,\theta)$ without knowing the demand of the uninformed, since their demand determines whether the price lies inside or outside the interval $\left(\underline{P}, \overline{P}\right)$. However, we do know from the lemma above that for any price $P \neq \underline{P}$, $m^U(P) \in \{0, 1-z\}$, and so the prices $\dfrac{w+z\mathbb{I}_{\theta=\overline{\theta}}}{x+1}$ and $\dfrac{w+z\mathbb{I}_{\theta=\overline{\theta}}+1-z}{x+1}$ both lie in $\widehat{P}(x,\theta)$ for the remaining values of x, where $\mathbb{I}_{\theta=\overline{\theta}}$ is an indicator function that equals 1 is $\theta = \overline{\theta}$ and zero otherwise. Finally, $\underline{P} \in \widehat{P}\left(x,\underline{\theta}\right)$ for $x+1 \in \left[\frac{w}{\underline{P}}, \frac{w+1-z}{\underline{P}}\right]$, since any $m^U\left(\underline{P}\right) \in [0, 1-z]$ is optimal. Likewise, $\underline{P} \in \widehat{P}\left(x,\overline{\theta}\right)$ for $x+1 \in \left[\frac{w+z}{\underline{P}}, \frac{w+1}{\underline{P}}\right]$.

Finally, to show that $\widehat{P}(x,\theta)$ is the minimal correspondence, one can easily show that any point in $\widehat{P}(x,\theta)$ can be a part of an equilibrium price function. In particular, for each distinct branch in $\widehat{P}(x,\theta)$, one can choose a price from the set $\widehat{P}(x,\theta)$ that uses that segment. Thus, using only five equilibria, one can construct an equilibrium using every element of $\widehat{P}(x,\theta)$. ■

The correspondence $\widehat{P}(x,\theta)$ is illustrated in Figure 3. To select an equilibrium price function $P(x,\theta)$ from $\widehat{P}(x,\theta)$, one has to make sure that the selection satisfies the restrictions imposed on any equilibrium price function $P(x,\theta)$ by Lemma 2 above. For example, if we choose $P\left(x,\underline{\theta}\right) = P'$ for some $P' \in \widehat{P}\left(x,\underline{\theta}\right)$, then P' must lie on the lower branch of

$\widehat{P}(x,\underline{\theta})$ if $P' \notin \text{range}\{P(\cdot,\overline{\theta})\}$, since P' would reveal that $\theta = \underline{\theta}$. Likewise, P' must lie in the upper branch of $\widehat{P}(x,\underline{\theta})$ if $P' \in \text{range}\{P(\cdot,\overline{\theta})\}$, since the expected return for any price $P' \in (\underline{P},\overline{P})$ such that $P' \in \{P(\cdot,\underline{\theta})\} \cap \text{range}\{P(\cdot,\overline{\theta})\}$ exceeds the return on money.

Graphically, we can easily confirm whether a candidate selection from $\widehat{P}(x,\theta)$ satisfies these restrictions by checking horizontally to see if a given price is contained in both $\text{range}\{P(\cdot,\underline{\theta})\}$ and $\text{range}\{P(\cdot,\overline{\theta})\}$, and then confirming whether the price is assigned to the correct branch of $\widehat{P}(\cdot,\theta)$ for each θ. The demand schedule for uninformed traders $x^U(P)$ can in turn be recovered from the equilibrium price function. Armed with this observation, it is easy to construct equilibrium price functions that, in contrast with the equilibrium constructed in Proposition 1, involve more than one point of discontinuity, and even a different number of points of discontinuity for the two values of θ. Likewise, it is easy to construct equilibria in which the demand of the uninformed is strictly positive even for prices below $\underline{P}$, since such prices fully reveal that fundamentals are favorable, again in contrast with Proposition 1. Thus, neither the demand of uninformed agents nor the equilibrium price function are uniquely determined in equilibrium. However, since $\widehat{P}(x,\theta)$ exhibits a fold catastrophe for both values of θ, it follows as an immediate corollary that in any rational expectations equilibrium, there must exist at least one point of discontinuity for each realization of θ.

Corollary: in any rational expectations equilibrium, the equilibrium price function $P(x,\theta)$ for either θ must be discontinuous at some value of x.

Crashes driven by panic-type behavior among uninformed traders are thus inherent to the economy we describe, and will occur in any equilibrium in this market. However, the fact that there are multiple rational expectations equilibria implies that the magnitude of crashes, the number of crashes, and the prices at which crashes occur are not uniquely pinned down.

3. Discussion

Now that we have constructed a discontinuous price function as an equilibrium phenomenon, we can use it to gain some insights about the nature of stock market crashes that are driven by uninformed traders. We begin with some observations about general properties of equilibrium price functions. We then offer some comments as to whether these observations are likely to survive under alternative assumptions, including dynamic extensions.

3.1. Properties of Stock Market Crashes

To better understand what determines the likelihood, magnitude, and nature of price crashes in asymmetric information environments, we can carry out various comparative static exercises on the equilibrium set $\widehat{P}(x, \theta)$ depicted in Figure 3. Several conclusions emerge from our model.

Observation 1: Crashes can only occur if both informed and uninformed traders are present. Obviously, uninformed traders are essential for generating a crash that is due to a rational panic, since they are the ones who have an upward sloping demand curve. Formally, as $z \to 1$, taking limits on the equilibrium price correspondence $\widehat{P}(x, \theta)$ described in Proposition 2 yields

$$\widehat{P}(x, \underline{\theta}) = \frac{w}{x+1}$$

$$\widehat{P}(x, \overline{\theta}) = \begin{cases} \dfrac{w}{x+1} & \text{for} \quad x+1 < \dfrac{w}{\overline{\theta}} \\ \overline{\theta} & \text{for} \quad x+1 \in \left[\dfrac{w}{\overline{\theta}}, \dfrac{w+1}{\overline{\theta}}\right] \\ \dfrac{w+1}{x+1} & \text{for} \quad x+1 > \dfrac{w+1}{\overline{\theta}} \end{cases}$$

Hence, $\widehat{P}(x, \theta)$ collapses from a correspondence to a continuous function. Since this price function is continuous, crashes cease to exist in the absence of uninformed agents. This is not surprising, since the crash is driven entirely by the mass withdrawal of uninformed traders from the market. However, informed agents are similarly crucial for generating a crash, since the upward sloping demand curve among uninformed traders arises only because uninformed agents seek to extract information from prices when prices convey useful information. Formally, as $z \to 0$, the correspondence $\widehat{P}(x, \theta)$ converges to

$$\widehat{P}(x, \cdot) = \begin{cases} \dfrac{w}{x+1} & \text{for} \quad x+1 < \dfrac{w}{\overline{\theta}} \\ \rho\overline{\theta} & \text{for} \quad x+1 \in \left[\dfrac{w}{\rho\overline{\theta}}, \dfrac{w+1}{\rho\overline{\theta}}\right] \\ \dfrac{w+1}{x+1} & \text{for} \quad x+1 > \dfrac{w+1}{\rho\overline{\theta}} \end{cases}$$

Technically, the two cutoffs $\overline{P}$ and $\underline{P}$ converge to $\rho\overline{\theta}$ and 0, respectively, as $z \to 0$. Once again, the correspondence $\widehat{P}(x, \theta)$ collapses to a continuous function, and there exists a unique equilibrium price function given by $\widehat{P}(x, \theta)$ above. Thus, crashes cease to exist in the absence of informed agents. The lower cutoff $\underline{P}$ which generates the non-monotonicity becomes non-binding as $\underline{P} \to 0$ since prices are always strictly positive. The demand curve of the uninformed thus converges to a strictly downward sloping curve, and the associated discontinuity in the equilibrium price function disappears.

Observation 2: Crashes can occur regardless of the value of the underlying asset. The existence of crashes is not predicated on being in a particular state of the world θ, i.e. crashes can occur even when the underlying asset is inherently valuable. In fact, since the set $\widehat{P}\left(x, \overline{\theta}\right)$ in Proposition 2 is folded over itself, crashes necessarily occur when the fundamentals are favorable in our model. This observation is important, since it distinguishes our model from alternative explanations such as the ones we cite in the Introduction. For example, the informational view advanced by Kraus and Smith [12] and subsequent authors argues that crashes occur when small changes reveal that fundamentals are unfavorable, causing agents to change their portfolio allocations and precipitate a large drop in the price. Under this view, crashes should only occur when agents learn that fundamentals are unfavorable. The same is true of the Madrigal and Scheinkman [16] model, since the information set is coarsened only when asset fundamentals are unfavorable. Although our model implies crashes will occur in both states of the world, the magnitude of a crash will not necessarily be the same in the two states of the world. For example, for the equilibrium identified in Proposition 1, the percentage decline in the price when $\theta = \overline{\theta}$ is given by

$$\frac{P^{*} - \underline{P}}{\underline{P}} = -\frac{1-z}{1+w}$$

while the percentage decline in price when $\theta = \underline{\theta}$ is given by

$$\frac{P^{*} - \frac{1-z+w}{w}P^{*}}{\frac{1-z+w}{w}P^{*}} = -\frac{1-z}{1-z+w}$$

which are unequal as long as the number of informed traders $z \neq 0$. This suggests that in a dynamic setting where agents get to observe prices over time, agents might be able to learn something about the fundamentals ex-post by observing the size of a crash, a point we return to below when we discuss extensions to dynamic environments.

The ability of our model to generate crashes even when the underlying asset is valuable should serve to dispel a common misperception that crashes are inherently associated with the bursting of speculative bubbles on underlying assets, i.e. that crashes serve to correct discrepancies between the price of an asset and its fundamental value. This view accurately captures what happens in the bad state of the world in our model; the asset is inherently worthless, and a crash reduces the price of the asset towards its fundamental value of zero. But in the good state of the world, a crash drives the price of the asset further away rather than toward its fundamental value of $\overline{\theta}$. Hence, crashes could just as well precipitate deviations from the fundamental value of an asset as eliminate them. Intuitively, since crashes arise when uninformed traders panic and pull out of the market, there is no reason that they would panic only when the fundamentals – which they are ignorant of – happen to be favorable. Empirically, there is suggestive evidence that the 1987 crash occurred despite the fact that the fundamentals were sound, providing support for models in which crashes can occur independently of underlying asset values. For example, Eichengreen [7], in comparing the 1929 and 1987 crashes, concludes that "probably the crucial difference between the two episodes was the state of the economy immediately preceding the crash. In the first nine months of 1987, spending was strong. In October 1929, in contrast, a full-blown recession was already under way." (p247)

Observation 3: The magnitude of a crash depends on the number of uninformed and naive traders in the market. As already remarked above, the magnitude of the crash can vary with the underlying state of the world. The equilibrium we present in Proposition 1 suggests crashes will tend to be more severe in the bad state of the world. This is quite intuitive: in the good state of the world, informed agents are willing to buy up assets that uninformed traders reject, thereby limiting the decline in price, at least to the extent afforded to them given their limited wealth. However, there do exist pathological selections from the correspondence $\widehat{P}(x, \theta)$ in Proposition 2 in which the percentage price decline associated with some price jump when $\theta = \overline{\theta}$ is greater than for some price jump that could occur when $\theta = \underline{\theta}$. It can be shown that such an outcome can be ruled out by restricting attention to equilibria where the price $P(x, \theta)$ is weakly decreasing in x, i.e. equilibria in which, intuitively, an increase in the effective supply of assets x lowers the price of the asset.

Similarly, the magnitude of a crash tends to increase with the wealth available to uninformed traders $1 - z$. This is true for the equilibrium price function described in Proposition 1, as well as any other equilibrium price function in which a crash amounts to a jump from the very upper branch of $\widehat{P}(x, \theta)$ to the very lower branch of $\widehat{P}(x, \theta)$. Intuitively, the discontinuity arises because we must inevitably reach a point at which uninformed traders shift all of their wealth from stocks into money, leading to a decline in the price of the asset. The more wealth uninformed traders have invested in the stock market, the more the price will fall when a crash occurs. The magnitude of the crash will also tend to be larger in these same equilibria if the amount of wealth that comes from non-sophisticated trade w is small. This too is because passive investors help to prop up stock prices by buying up stocks from uninformed agents when the latter dump off their stock. Thus, our model can capture the popular notion that passive traders who follow simple strategies that are not very responsive to price help to stabilize markets, while active (but poorly informed) traders who try to act strategically, e.g. day-traders, destabilize them.

Observation 4: Prices need not convey information monotonically. Although low prices are generally more likely to convey negative information about the asset, the expectation $E[\theta \mid P(\cdot,\cdot) = P]$ in (2.6) need not be monotonic in P. For example, for the equilibrium described in Proposition 1, there is an intermediate region of prices $\left(\underline{P}, \frac{w+1-z}{w}P^*\right]$ in which prices are fully revealing in the good state of the world. This occurs because the sensitivity of demand to prices is different in the two states of the world; a decrease in price triggers a larger increase in demand in the good state when informed traders are active than in the bad state of the world when they are not. As a result, it is impossible to construct a price function in which the magnitude of the crash is the same in both states of the world. Hence, when $\theta = \overline{\theta}$, prices just above the crash point are fully revealing, while prices just below the crash point are not. This result is somewhat noteworthy, since it provides a contrast to models in which crashes occur when information is revealed through small changes in the fundamentals. In those models, prices fall sharply in response to the arrival of clear but negative information. Here, a shift from the pre-crash price to the post-crash price need not be associated with any resolution of uncertainty, and might even be associated with an increase in uncertainty. However, this pattern does not extend generally across all possible equilibrium price functions that can be constructed from the set $\widehat{P}(x, \theta)$, and, as we discuss below, may not survive in dynamic contexts.

3.2. Robustness

Given that we are forced to rely on strong assumptions regarding preferences, distributions, and the trading strategies available to agents (e.g. no short sales) to solve the model in closed-form, it is natural to ask whether our model and the observations it implies represent a technical curiosity, or whether they point to a more common phenomenon that can hold under more general assumptions. Since solving for equilibrium in models with upward sloping demand schedules is computationally challenging, it is hard to assess whether a crash would arise under a particular set of alternative assumptions that are arguably more plausible than the ones we consider. However, the model can still shed light on which features are important for generating crashes as an equilibrium phenomenon, and thus how robust our results are likely to be in alternative environments.

Recall that crashes arise in our model because the set of market-clearing prices $\widehat{P}(x, \theta)$ is folded over itself. This in turn is due to the backwards-bending demand for assets among uninformed traders, i.e. to the fact that their demand for assets is upward sloping at low prices but downward sloping at high prices. There are two features in our model that ensure the demand of uninformed traders is downward sloping at high prices. First, we assume the support of the distribution of θ is bounded. Hence, if prices ever rise above $\overline{\theta}$, the demand for the asset will fall to zero. Second, we prevent agents from selling money short. Consequently, a high enough price will exhaust the resources of agents and leave them unable to continue purchasing the same amount of risky assets. Both of these assumptions are ruled out in the models of Admati [1] and Wang [20] that generate globally upward-sloping demand curves; this is why agents continue to increase their demand for assets even at very high prices. Introducing either feature into these models could potentially give rise to backwards bending demand for risky assets; after all, the intuition for upward sloping demand in these model does not rest on what happens to demand at very high prices. However, introducing this modification would render such models virtually impossible to solve.

Next, several distributional assumptions in our model ensure the demand of uninformed traders is upward sloping at lower prices, or, more specifically, that the uninformed will be willing to buy risky assets at intermediate prices but not at very low prices. The reason uninformed agents are willing to purchase the asset at intermediate prices is that we assume the probability of favorable fundamentals ρ is large. Recall from our previous discussion that for a given price P, the conditional expectation $E[\theta \mid P(x,\theta) = P]$ converges to $\overline{\theta}$ as $\rho \to 1$. Thus, for high enough ρ, a risk-neutral trader would strictly prefer the risky asset to money. In fact, if ρ is close enough to 1, the risk associated with the asset becomes negligible, so an uninformed risk-averse agent would also invest a positive amount of his wealth in the risky asset. Hence, the key assumption that leads to positive demand for the asset in some intermediate price range is that bad outcomes are relatively rare.

Two additional features of our model imply that demand will drop to zero at prices below this intermediate range. First, we assume that the lowest possible payoff $\underline{\theta}$ on the asset is zero. Thus, the asset may not be attractive even when it is virtually free. Second, the density function for x decays at an exponential rate, ensuring the likelihood ratio $\phi(P)$ explodes to infinity as $P \to 0$ fast enough to satisfy condition (2.8). Intuitively, the exponential distribution insures that the level of noise trading required to offset the high demand of informed traders in the good state of the world is much less likely than the level of noise trading required in the bad state of the world. Together, these two assumptions imply that at prices close to zero, an uniformed agent will assign a very low conditional expectation to θ, even less than the price of the asset P. Any uninformed trader, whether risk-averse or risk-neutral, will therefore strictly prefer to invest in money. Thus, what leads to a fall in the demand for assets at lower prices is the possibility of a very bad outcome combined with the fact that only a very unlikely amount of noise trade can clear the market when informed agents are aware this bad outcome has not been realized.

In general, upward sloping demand is likely to arise under various alternative assumptions on the distribution of market fundamentals. In particular, if $\frac{\partial}{\partial P} E[\theta \mid P(\cdot,\cdot) = P] \gg 1$ at those prices where uninformed agents invest a positive amount of wealth in the risky asset, the conditional expectation of θ will fall faster than the price of the asset. Eventually, then, the expected return on the risky asset will fall below the return on money, potentially at some price that is far away from $P = 0$. Thus, our results do not hinge on properties that govern the tail distribution of x, but could arise for whenever the distribution of

x concentrates enough mass on particular ranges of x in a way that uninformed agents anticipate the bad outcome is far more likely when they observe a certain price. Since upward sloping demand can easily arise in asymmetric information models with goods of varying quality, as evidenced by the weak conditions provided by Wilson [22, 23] for upward sloping demand, it seems likely that the same would be true for trade in financial assets. Unfortunately, we are unable to prove this formally. The reason is that in noisy rational expectations equilibria, the conditional expectation of θ depends not only on the observed price P but also on the demand of uninformed traders, which determines the amount of noise trade $x(P, \underline{\theta})$ necessary to clear the market in the bad state and which enters into the likelihood ratio $\phi(P)$. By contrast, in Wilson's framework the conditional expectation of θ does not depend on the demand side of the economy, and thus is easier to compute. But this represents more of a computational challenge than a conceptual one; the existence of upward sloping demand should still depend on how much weight the underlying distribution gives to states of the world associated with high returns compared with states associated with low returns.

To summarize, the backwards bending demand for assets among uninformed traders arises from rather plausible features of our model, although we have to impose additional assumptions to be able to solve the model in closed form. As long as the fraction of wealth that belongs to informed agents z is sufficiently small, the aggregate demand curve will inherit the properties of the demand of uninformed agents. Provided informed and noise traders exhibit downward sloping demand for the asset, the aggregate demand schedule will then fold over itself, producing the same catastrophe that leads to discontinuous equilibrium price functions as under our particular assumptions.

3.3. Dynamics

So far, our discussion has focused on static environments. But crashes are inherently dynamic phenomena that play out over time. Recall that the definition of a crash in our static economy intended to capture the notion that in the real world, stock prices appear to fall sharply without any corresponding changes in the underlying fundamentals. In other words, crashes are price declines that are not associated with large changes in the fundamentals associated with an asset, as opposed to more conventional price declines that are triggered by a change in the fundamentals, e.g. a large fall in present or future dividends or a sudden increase in demand for liquidity. Formally, in a discrete-time setting, we will say that an asset is vulnerable to a crash if for any real number N, there exists a realization for the path of fundamentals $\{\Theta_t\}$ and a path of equilibrium prices $\{P_t\}$ consistent with these fundamentals for which the ratio of the change in the equilibrium price of the asset $\|P_t - P_{t-1}\|$ to the change in fundamentals $\|\Theta_t - \Theta_{t-1}\|$ exceeds N at some date t. In other words, changes in prices can be arbitrarily larger than the corresponding changes in fundamentals. If we let the length of a period go to zero, a crash would imply that there exists a realization for the path of fundamentals $\{\Theta_t\}$ that is continuous with respect to time in some interval $[t, t+\Delta)$ and a corresponding path $\{P_t\}$ for the equilibrium price of the asset that is consistent with these fundamentals and which is not continuous with respect to time over this same interval. The question is whether our results above, which establish the existence of static equilibrium price functions that are discontinuous *with respect to the fundamentals* $\Theta_t = (x_t, \theta_t)$, are useful for modelling crashes in dynamic environments where the path of prices must be discontinuous *with respect to time* for a continuous path of the fundamentals.

One way to use the discontinuity in our static model to construct a dynamic crash is to suppose individuals live for only one period and then sell their assets to the next generation of traders, who, other than noise traders, are each endowed with one unit of money. The supply of the risky asset in each period is just the amount sold by agents who owned the stock in the previous period right before they die. The payoff to buying a risky asset is $\theta_t + P_{t+1}$, i.e. the dividend θ_t plus the capital gain of selling the asset to the next generation. If θ_t and x_t are i.i.d. random variables, we can look for equilibria in which the price function

$P(x,\theta)$ is constant over time, since there are no other state variables that are relevant for agents. In that case, all agents form the same expectation over the future price of the asset $E_{x,\theta}[P(x_{t+1},\theta_{t+1})]$. Since this expectation is just a constant, the analysis reduces to a static analysis in each period with payoffs $\widehat{\theta} = \theta + E[P(x_{t+1},\theta_{t+1})]$. The main difference between this dynamic model and the static model above is that the payoff in the bad state of the world $\underline{\theta}$ is now strictly positive, since $E[P(x_{t+1},\theta_{t+1})] > 0$. However, it is easy to show that the equilibrium price function in the static model remains discontinuous as long as $\underline{\theta}$ is sufficiently close to zero. We can always ensure that the expectation of the price $E[P(\cdot,\cdot)]$ is arbitrarily small by setting μ to be sufficiently large, so that our previous analysis applies. If we then choose a path for $\{\Theta_t\}$ where θ_t is constant and x_t traverses the point of discontinuity of $P(\cdot,\cdot)$, we can generate a fall in the price of the asset that far exceeds the change in fundamentals over the same period.

In this example, agents maximize one-period profits and the vector of fundamentals Θ_t is independent across time. Thus, our example is essentially a sequence of static economies, and so it is not surprising we can use the discontinuity of the static price function with respect to Θ to construct a price path that is essentially discontinuous with respect to time. A more interesting question is whether our results are useful for understanding crashes in truly dynamic environments, where agents have long horizons and the fundamentals follow more conventional stochastic processes. Constructing such a model, much less solving it, is beyond the scope of this paper. However, we briefly offer some comments on complications that can arise in dynamic environments.

To be concrete, suppose an asset pays a dividend θ_t each period, where θ_t can now assume multiple values rather than only two. Suppose further that the vector of fundamentals $\Theta_t = (x_t,\theta_t)$ follows a discrete-time Markov process. Since the definition of a crash involves the behavior of equilibrium prices along a continuous realization of the fundamentals, it seems natural to focus on Markov processes in which the realizations of the process tend to continuous sample paths in the limit as the length of a period collapsed to zero. Note that our assumption above that fundamentals are independent over time implies this process would not tend to continuous sample paths, since by construction there is nothing that ties the value of Θ in period t to be close to its value in period $t-1$. Agents know the Markov process that governs Θ_t and decide how much of their wealth to allocate between money and the risky asset each period. Informed agents optimize given the history $\{\Theta_s\}_{s=0}^{t}$, while uninformed agents get to observe only the history of prices $\{P_s\}_{s=0}^{t}$.

In trying to use discontinuities in the static price function $P(\Theta)$ to generate dynamic crashes, one potential problem is that if the function $P(\cdot)$ does not depend on time and the fundamentals are essentially continuous, a discrete price jump could only occur in the region where $P(\Theta)$ is discontinuous. If the magnitude of a crash and the price at which a crash occurs varies with current dividends θ_t, as our static model suggests is likely, agents would learn θ_t (and hence x_t) by observing the crash. But then a crash could no longer arise in equilibrium when dividends are favorable, since agents would immediately learn this and repurchase the asset, bidding up its price and denying the possibility that the price in the wake of a crash could remain an equilibrium. This would destroy what we view as the unique aspect of our model compared with previous work, namely that crashes are driven by relatively uninformed agents who act out of ignorance; if the crash tends to reveal the underlying state to any previous uninformed traders, the model would reduce to models described in previous work where crashes are driven by the sudden revelation of information. Thus, in order to use the discontinuity of the static price $P(\Theta)$ to generate dynamic crashes, we would have to appeal to a more complicated environment in which a crash does not reveal the exact state of the world. For example, if the vector of fundamentals involved a more detailed description of future dividends stream, e.g. information both on current dividends as well expected dividend growth, uninformed traders might no longer be able to infer Θ_t perfectly from the magnitude of the crash or the price at which it occurred. Alternatively, we could allow for a jump component in the stochastic process governing fundamentals, so that uninformed agents would be unable to distinguish whether a decline in price is precipitated by their panic or by informed traders responding to an abrupt change in fundamentals.

Another potential problem with using the discontinuity of the static price function $P(\Theta)$ to generate dynamic crashes is that if agents know the static equilibrium price function and realize that the fundamentals are continuous, they might be reluctant to hold the asset at a price just above the crash level fearing a large capital loss. But if that were the case, the original price could not be an equilibrium. This concern is mitigated by the fact that crashes can occur at different price levels for different values of the dividend θ_t, as is the case in the equilibrium in Proposition 1. If there were a continuum of values for θ_t, each of which is associated with a crash at a different price, the risk of a crash at any particular price might not be large enough to discourage uninformed agents from holding on to the asset. The demand of informed traders is more complicated, since they observe Θ_t and are aware of whether crashes are likely, and so they might choose to stay out of the market in regions close to a point of discontinuity. It is therefore not obvious whether the equilibrium could still hang together when the fundamentals evolve continuously. However, once again we could introduce a jump component into the fundamentals, which could provide incentive for informed agents to hold on to the asset even at prices that are near the threshold of a crash in the hope of a reaping large potential capital gain with some probability. In this case, the model could still yield paths of $\{\Theta_t\}$ with corresponding equilibrium prices $\{P_t\}$ where the size of the price change $||P_t - P_{t-1}||$ is "very large" compared to the changes in fundamentals $||\Theta_t - \Theta_{t-1}||$.

This discussion suggests that the static notion of crashes we illustrate could potentially be used to construct crashes in dynamic models. However, given the inherent difficulties described above, as well as the complicated portfolio allocation problem that would inevitably arise when agents face longer horizons, more work is necessary before we can develop a satisfactory dynamic model of crashes that progresses beyond static models.

4. Conclusion

This paper constructs a model that formalizes the notion that crashes – abrupt changes in the price of a stock that occur despite the absence of any corresponding change in the economic fundamentals of the underlying asset – could arise because of the behavior of uninformed traders who panic and cause the price of stocks to plummet. The mechanism we describe is identical to the one developed by Gennotte and Leland [8] in which exogenous portfolio insurers precipitate price declines by selling stocks when prices fall, thus magnifying small price changes into large, discontinuous jumps. In our framework, it is uninformed traders who sell stocks at low prices, but this is rational for them since they try to read information they know is available to other traders who are active in the market. Thus, there is no need to appeal to exogenous trading motives or systematic misperception among traders about the motives of others in order to sustain this mechanism for crashes.

While we view our model as a theoretical contribution of how this explanation can be formally modelled, we believe our story has some empirical plausibility for actual stock trading, and so we conclude with short remarks why our explanation can accord with actual crashes. Previous authors have ascribed as an important role to uninformed traders in precipitating price crashes. For example, White [21] writes

> During the tulipmania, Garber notes that the middle classes and even monied workers began to speculate in the market for tulips, which previously had been the province of specialists. In their enthusiasm to participate, new investors seem incredibly incautious... In 1873, new German investors played a prominent role. They busily acquired new, untested foreign securities for their portfolios. In the 1920s the shift in business financing from short-term commercial bank loans to bonds and stocks meant that instead of commercial banks who had considerable experience in evaluating firms, the general investing public became the chief creditors of corporations. Americans who had never owned stocks before were now buying. Given the increased difficulty of evaluating fundamentals and the general optimism from the decade of prosperity, it is not surprising that prices were pushed above fundamentals. Similarly in the 1980s, new financial instruments drew in new investors from both at home and abroad. (p238)

The role of nervous investors was even noted in the original Brady report [4] as one of the causes of the 1987 crash. As quoted by Wyatt [24], the report notes that "to the market, their [mutual fund investors'] behavior looked much like that of the portfolio insurers, that is, selling without primary regard to price." Our paper is the first to our knowledge to provide a formal model that rationalizes these observations. Given that stock market participation has increased over the past decade, particularly among small investors who are more likely to be uninformed about economic fundamentals in real time, our analysis suggests that stock markets today may be vulnerable to crashes, despite the fact that the use of stocks for portfolio insurance has largely disappeared. Moreover, the recent shift from passive investing strategies to more aggressive trading practices such as day-trading, which are more reactive to prices, might very well serve to magnify the magnitude of price crashes if they were to occur, since passive traders tend to mitigate price crashes by blindly buying up the stocks from those who panic and sell them.

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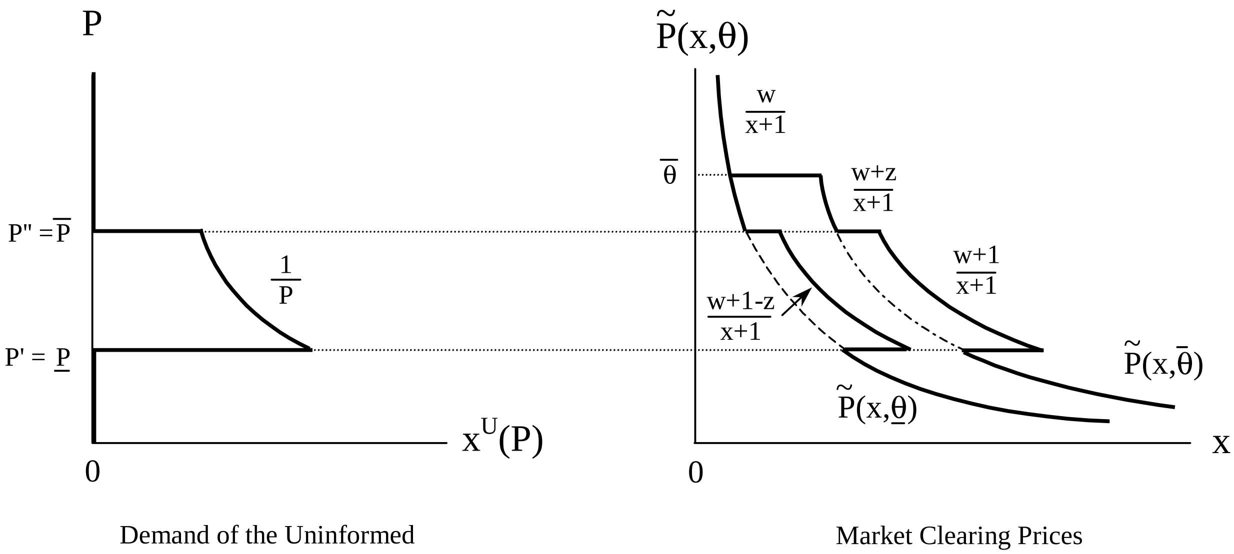

Figure 1: Conjectured Demand and Associated Market Clearing Prices

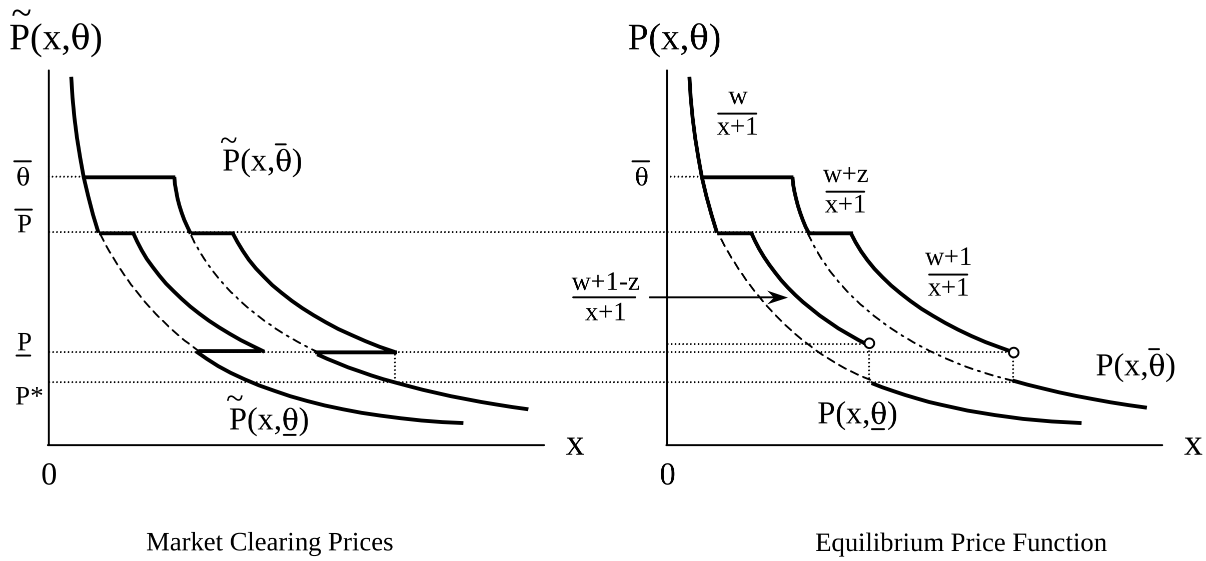

Figure 2: Constructing an Equilibrium Price Function

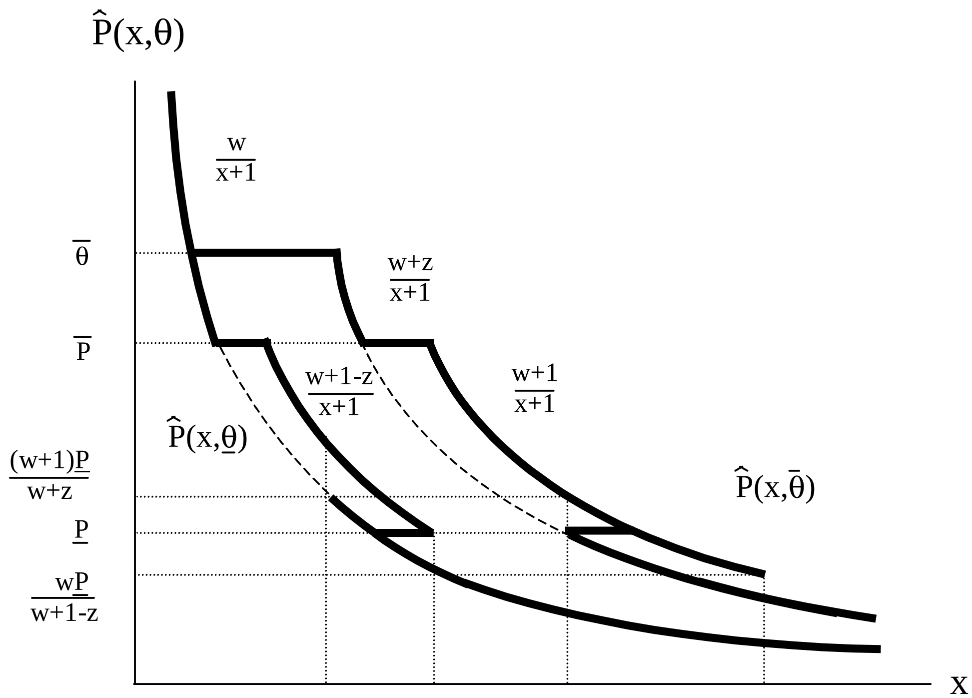

Figure 3: The Set of Equilibrium Price Functions