Internet Appendix to Accompany Leverage

Tano Santos and Pietro Veronesi

This Internet appendix to Santos and Veronesi (2020) contains the following additional material:

1. Section IA1. Proofs to propositions and corollaries.
2. Section IA2. Estimating systematic consumption volatility.
3. Section IA3. Model's implications for other parameter choices.
4. Section IA4. Households' Leverage: Robustness.

IA1. Proofs

Proof of Proposition 1. Starting with the maximization problem in equation (7), the Lagrangean for the static maximization is

$$L = E_0 \left[\int_0^\infty e^{-\rho t} \log \left(C_t - \psi_{it} Y_t \right) dt \right] - \frac{1}{\phi_i} E_0 \left[\int_0^\infty M_t C_t dt - w_i M_0 \right]$$

where ϕ_i is the inverse of the Lagrange multiplier. The first order conditions are

$$e^{-\rho t} \frac{1}{C_{it} - \psi_{it} Y_t} = \frac{1}{\phi_i} M_t$$

which give

$$C_{it} - \psi_{it} Y_t = \phi_i e^{-\rho t} M_t^{-1} \tag{IA.1}$$

Aggregate across i to obtain

$$\int C_{it} di - \int \psi_{it} di Y_t = \int \phi_i di \; e^{-\rho t} M_t^{-1}$$

Normalize without loss of generality $\int \phi_i di = 1$, use $\psi_{it} = \gamma_i - \gamma_i I_t^{-1}$ and impose the market clearing condition $\int C_i di = Y_t$ to obtain

$$Y_t - \left(\int \gamma_i di - \int \gamma_i di I_t^{-1} \right) Y_t = e^{-\rho t} M_t^{-1}$$

or

$$Y_t I_t^{-1} = e^{-\rho t} M_t^{-1}$$

Solving for M_t we obtain the equilibrium state price density:

$$M_t = e^{-\rho t} Y_t^{-1} I_t \; .$$

Substituting the state price density back in the consumption equation (IA.1) provides the optimal consumption policy

$$\begin{aligned} C_{it} &= \phi_i e^{-\rho t} M_t^{-1} + \psi_{it} Y_t \\ &= \left[\phi_i I_t^{-1} + \psi_{it}\right] Y_t \\ &= \left[\phi_i I_t^{-1} + \gamma_i - \gamma_i I_t^{-1}\right] Y_t \end{aligned}$$

As $I_t > 1$ and $\phi_i > 0$, we have $C_{it} > 0$ for all t and and for all i. We can thus write

$$C_{it} = \left[\gamma_i + (\phi_i - \gamma_i) I_t^{-1}\right] Y_t \tag{IA.2}$$

We now solve for the Lagrange multiplier $1/\phi_i$ by using the budget constraint at time 0. Wealth at any time t is the present value of future optimal consumption and it is thus given by

$$\begin{aligned} W_{it} &= M_t^{-1} E\left[\int_t^\infty M_\tau C_{i\tau} d\tau\right] \\ &= Y_t I_t^{-1} E\left[\int_t^\infty e^{-\rho(\tau-t)} Y_\tau^{-1} I_\tau C_{i\tau} d\tau\right] \\ &= Y_t I_t^{-1} E\left[\int_t^\infty e^{-\rho(\tau-t)} I_\tau \left[\gamma_i + (\phi_i - \gamma_i) I_t^{-1}\right] d\tau\right] \\ &= Y_t I_t^{-1} \left[E\left[\int_t^\infty e^{-\rho(\tau-t)} I_\tau \gamma_i d\tau\right] + E\left[\int_t^\infty e^{-\rho(\tau-t)} I_\tau (\phi_i - \gamma_i) I_t^{-1} d\tau\right]\right] \\ &= Y_t I_t^{-1} \left[\gamma_i \int_t^\infty e^{-\rho(\tau-t)} \left[\overline{I} + \left[I_t - \overline{I}\right] e^{-k(\tau-t)}\right] d\tau + \int_t^\infty e^{-\rho(\tau-t)} (\phi_i - \gamma_i) d\tau\right] \\ &= Y_t I_t^{-1} \left[\gamma_i \left[\frac{\overline{I}}{\rho} + \frac{\left[I_t - \overline{I}\right]}{\rho + k}\right] + \frac{(\phi_i - \gamma_i)}{\rho}\right] \\ &= Y_t I_t^{-1} \left[\left(\frac{\overline{I}\gamma_i + (\phi_i - \gamma_i)}{\rho} - \frac{\gamma_i \overline{I}}{\rho + k}\right) + \frac{\gamma_i I_t}{\rho + k}\right] \\ &= Y_t \left[\left(\frac{\overline{I}\gamma_i (\rho + k) + (\phi_i - \gamma_i)(\rho + k) - \rho \gamma_i \overline{I}}{\rho(\rho + k)}\right) I_t^{-1} + \frac{\gamma_i}{\rho + k}\right] \end{aligned}$$

or

$$W_{it} = Y_t \left[\left(\frac{\overline{I}\gamma_i k + (\phi_i - \gamma_i)(\rho + k)}{\rho(\rho + k)}\right) I_t^{-1} + \frac{\gamma_i}{\rho + k}\right] \tag{IA.3}$$

As of time zero, wealth is thus given by

$$W_{i0} = M_0^{-1} E\left[\int_0^\infty M_\tau C_{i\tau} d\tau\right]$$

or

$$W_{i0} M_0 = E\left[\int_0^\infty M_\tau C_{i\tau} d\tau\right]$$

We can thus impose the initial endowment condition

$$w_i = W_{i0}$$

and solve the following equation for ϕ_i

$$w_i = W_{i0} = Y_0 \left[\left(\frac{\overline{I}\gamma_i k + (\phi_i - \gamma_i)(\rho + k)}{\rho(\rho + k)} \right) I_0^{-1} + \frac{\gamma_i}{\rho + k} \right]$$

In particular, assume that at time 0 the economy is at the stochastic steady state

$$I_0 = \overline{I}$$

so that

$$w_i = Y_0 \left[\frac{\gamma_i k + (\phi_i - \gamma_i)(\rho + k)\,\overline{I}^{-1}}{\rho(\rho + k)} + \frac{\rho\gamma_i}{\rho(\rho + k)} \right]$$

or

$$w_i = Y_0 \left[\frac{\gamma_i + (\phi_i - \gamma_i)\,\overline{I}^{-1}}{\rho} \right]$$

Assume without loss of generality that

$$Y_0 = \rho$$

to obtain

$$\phi_i = \frac{w_i + \gamma_i \overline{I}^{-1} - \gamma_i}{\overline{I}^{-1}} \tag{IA.4}$$

This proves claim (a) in Proposition 1. Note that

$$\int \phi_i di = \frac{\int w_i di + \int \gamma_i di \overline{I}^{-1} - \int \gamma_i di}{\overline{I}^{-1}} = \frac{1 + \overline{I}^{-1} - 1}{\overline{I}^{-1}} = 1$$

In addition, the constraint

$$\phi_i > 0$$

is satisfied if and only if

$$w_i + \gamma_i \overline{I}^{-1} - \gamma_i > 0$$

or

$$w_i > \gamma_i - \gamma_i \overline{I}^{-1}$$

which is condition (A1).

Substitute ϕ_i back into optimal consumption (IA.2) to obtain

$$\begin{aligned} C_{it} &= \left[\gamma_i + (\phi_i - \gamma_i) I_t^{-1}\right] Y_t \\ &= \left[\gamma_i + \left(\frac{w_i + \gamma_i \overline{I}^{-1} - \gamma_i}{\overline{I}^{-1}} - \gamma_i \right) I_t^{-1} \right] Y_t \\ &= \left[\gamma_i + (w_i - \gamma_i) \frac{\overline{I}}{I_t} \right] Y_t \end{aligned}$$

which is claim (b) of Proposition 1.

Proof of Propositions 2 and 3. From (IA.3) the stock price thus indeed satisfies

$$\begin{aligned} P_t &= \int W_{it} di \\ &= Y_t \left[\left(\frac{\bar{I} \int \gamma_i di k + \left(\int \phi_i di - \int \gamma_i di \right) (\rho + k)}{\rho (\rho + k)} \right) I_t^{-1} + \frac{\int \gamma_i di}{\rho + k} \right] \\ &= Y_t \left[\left(\frac{\bar{I} k + (1 - 1)(\rho + k)}{\rho (\rho + k)} \right) I_t^{-1} + \frac{1}{\rho + k} \right] \\ &= Y_t \frac{1}{(\rho + k)} \left[\frac{k \bar{I} S_t}{\rho} + 1 \right] \\ &= Y_t \frac{\rho + k \bar{I} S_t}{(\rho + k)\rho} \end{aligned}$$

which is the claim in Proposition 2 (see also Menzly, Santos, and Veronesi (2004)).

The stochastic discount factor follows from Ito's lemma applied to the state price density, giving

$$\begin{aligned} \frac{dM_t}{M_t} &= -\rho dt + I_t^{-1} dI_t - \frac{dY}{Y_t} + \left(\frac{dY}{Y} \right)^2 - \frac{dY_t}{Y_t} \frac{dI_t}{I_t} \\ &= -\rho dt + I_t^{-1} k \left(\bar{I} - I_t \right) dt - v \sigma_Y (I_t) dZ_t - \mu_Y dt - \sigma_Y (I_t) dZ_t + \sigma_Y (I_t)^2 dt \\ &\quad + v \sigma_Y (I_t)^2 dt \\ &= -\left(\rho + \mu_Y - (1 + v) \sigma_Y (I_t)^2 + k \left(1 - \bar{I} S_t \right) \right) dt - (1 + v) \sigma_Y (I_t) dZ_t \end{aligned}$$

That is

$$\begin{aligned} r_t &= \rho + \mu_Y - (1 + v) \sigma_Y (I_t)^2 + k \left(1 - \bar{I} / I_t \right) \\ \sigma_M &= (1 + v) \sigma_Y (I_t) \end{aligned}$$

which proves part (b) of Proposition 2 and Proposition 3.

Finally, to obtain the properties of returns, we first obtain the process for the surplus consumption ratio $S_t = 1/I_t$, given by

$$\begin{aligned} dS_t &= -I_t^{-2} dI_t + \frac{1}{2} 2 I_t^{-3} dI_t^2 \\ &= -I_t^{-2} \left[k \left(\bar{I} - I_t \right) dt - v I_t \sigma_Y (I_t) dZ_t \right] + \frac{1}{2} 2 I_t^{-3} \left[v^2 I_t^2 \sigma_Y (I_t)^2 \right] dt \\ &= -I_t^{-1} \left[k \left(\frac{\bar{I}}{I_t} - 1 \right) dt - v \sigma_Y (I_t) dZ_t \right] + I_t^{-1} \left[v^2 \sigma_Y (I_t)^2 \right] dt \\ &= S_t \left[k \left(1 - \bar{I} S_t \right) dt + v^2 \sigma_Y (I_t)^2 \right] + S_t v \sigma_Y (I_t) dZ_t \end{aligned}$$

where we denote for simplicity

$$\sigma_I(I) = v \sigma_Y(I) \tag{IA.5}$$

Thus, Ito's lemma implies that the stock return process is

$$\begin{aligned} dP_t &= \frac{\rho + k\bar{I}S_t}{(\rho+k)\,\rho} dY_t + \frac{k\bar{I}}{(\rho+k)\,\rho} Y_t dS_t + o\,(dt) \\ &= P_t \frac{dY_t}{Y_t} + P_t \frac{1}{\frac{\rho+k\bar{I}S_t}{(\rho+k)\rho} Y_t} \frac{k\bar{I}}{(\rho+k)\,\rho} Y_t dS_t + o\,(dt) \\ \frac{dP}{P} &= \frac{dY_t}{Y_t} + \frac{k\bar{I}}{\rho + k\bar{I}S_t} dS_t + o\,(dt) \end{aligned}$$

The diffusion term is therefore

$$\begin{aligned} \sigma_P\,(I_t) &= \sigma_Y\,(I_t) + \frac{k\bar{I}S_t v \sigma_Y\,(I_t)}{\rho + k\bar{I}S_t} \\ &= \left(1 + \frac{k\bar{I}/I_t v}{\rho + k\bar{I}/I_t}\right) \sigma_Y\,(I_t) \end{aligned}$$

and the expected return is

$$\begin{aligned} E\left[\frac{dP + Y dt}{P_t} - r_t dt\right] &= (1+v)\,\sigma_Y\,(I_t)\,\sigma_P\,(I_t) \\ &= (1+v)\left(1 + \frac{k\bar{I}/I_t v}{\rho + k\bar{I}/I_t}\right) \sigma_Y\,(I_t)^2 \end{aligned}$$

□

Proof of Proposition 4. Given the results of Propositions 1, 2, and 3, and the standard result that the efficient allocation maximize agents' utility, the only part left to show is the optimal allocation to stocks and bonds. From Cox and Huang (1989), the dynamic budget equation can be written as the present value of future consumption discounted using the stochastic discount factor. The optimal allocation can be found by finding the "replicating" portfolio, that is, the position in stocks and bonds that satisfies the static budget equation.

Consider agents' wealth obtained in (IA.3). Substituting ϕ_i from (IA.4)

$$\begin{aligned} W_{it} &= Y_t \left[\left(\frac{\bar{I}\gamma_i k + (\phi_i - \gamma_i)\,(\rho+k)}{\rho\,(\rho+k)}\right) I_t^{-1} + \frac{\gamma_i}{\rho+k}\right] \\ &= Y_t \left[\left(\frac{\bar{I}\gamma_i k + \left(\frac{w_i + \gamma_i \bar{I}^{-1} - \gamma_i}{\bar{I}^{-1}} - \gamma_i\right)(\rho+k)}{\rho\,(\rho+k)}\right) I_t^{-1} + \frac{\gamma_i}{\rho+k}\right] \\ &= Y_t \left[\frac{\gamma_i \rho + [(w_i - \gamma_i)\,\rho + w_i k]\,\bar{I}S_t}{\rho\,(\rho+k)}\right] \\ &= Y_t \frac{1}{\rho(\rho+k)} \left[\gamma_i \rho + (w_i(\rho+k) - \gamma_i \rho)\,\bar{I}S_t\right] \end{aligned}$$

From Ito's lemma, the diffusion of wealth process $dW_{i,t}/W_{i,t}$ is

$$\sigma_{W,i}(I_t) = \sigma_Y(I_t) + \frac{(w_i(\rho+k) - \gamma_i\rho)\,\overline{I}I_t^{-1}\sigma_I(I_t)}{\gamma_i\rho + (w_i(\rho+k) - \gamma_i\rho)\,\overline{I}I_t^{-1}} \tag{IA.6}$$

By market completeness (Cox and Huang (1989)), agent i's wealth is always equal to his allocation to stocks and bonds

$$W_{it} = N_{i,t}P_t + B_{it}$$

where B_{it} is the position in bonds (i.e. deposits or loans depending if positive or negative). From this latter expression, N_{it} must be chosen to equate the diffusion of the portfolio to the diffusion of wealth. That is, such that

$$N_{it}P_t\sigma_P(I_t) = W_{i,t}\sigma_{W,i}(I_t)$$

Solving for N_{it} gives

$$\begin{aligned}
N_{it} &= \frac{W_{it}\sigma_{Wi}(I)}{P_t\sigma_P(I)} \\
&= \frac{\left(\rho\gamma_i + (w_i(\rho+k) - \rho\gamma_i)\,\overline{I}/I_t\right)}{\left(\rho + k\overline{I}/I_t\right)} \left(\frac{\sigma_Y(I) + \frac{(w_i(\rho+k)-\rho\gamma_i)\overline{I}I_t^{-1}\sigma_I(I)}{\left(\rho\gamma_i + (w_i(\rho+k)-\rho\gamma_i)\overline{I}/I_t\right)}}{\sigma_Y(I) + \frac{k\overline{I}I_t^{-1}\sigma_I(I)}{\left(\rho+k\overline{I}/I_t\right)}} \right) \\
&= \frac{\left(\rho\gamma_i + (w_i(\rho+k) - \rho\gamma_i)\,\overline{I}/I_t\right)}{\left(\rho + k\overline{I}/I_t\right)} \left(\frac{\frac{\sigma_Y(I)\left(\rho\gamma_i + (w_i(\rho+k)-\rho\gamma_i)\overline{I}/I_t\right) + (w_i(\rho+k)-\rho\gamma_i)\overline{I}I_t^{-1}\sigma_I(I)}{\left(\rho\gamma_i + (w_i(\rho+k)-\rho\gamma_i)\overline{I}/I_t\right)}}{\frac{\sigma_Y(I)\left(\rho+k\overline{I}/I_t\right) + k\overline{I}I_t^{-1}\sigma_I(I)}{\left(\rho+k\overline{I}/I_t\right)}} \right) \\
&= \left(\frac{\sigma_Y(I)\left(\rho\gamma_i + (w_i(\rho+k) - \rho\gamma_i)\,\overline{I}/I_t\right) + (w_i(\rho+k) - \rho\gamma_i)\,\overline{I}I_t^{-1}\sigma_I(I)}{\sigma_Y(I)\left(\rho + k\overline{I}/I_t\right) + k\overline{I}I_t^{-1}\sigma_I(I)} \right) \\
&= \frac{\sigma_Y(I)\,\rho\gamma_i + \sigma_Y(I)\,\overline{I}/I_t\,(w_i(\rho+k) - \rho\gamma_i) + w_i(\rho+k)\,\overline{I}I_t^{-1}\sigma_I(I) - \rho\gamma_i\overline{I}I_t^{-1}\sigma_I(I)}{\sigma_Y(I)\left(\rho + k\overline{I}/I_t\right) + k\overline{I}I_t^{-1}\sigma_I(I)} \\
&= \gamma_i + (\rho+k)\,\frac{\sigma_Y(I)\,\overline{I}/I_t + \overline{I}I_t^{-1}\sigma_I(I)}{\sigma_Y(I)\left(\rho + k\overline{I}/I_t\right) + k\overline{I}I_t^{-1}\sigma_I(I)}\,(w_i - \gamma_i) \\
&= \gamma_i + (\rho+k)\,\frac{\sigma_Y(I)\,\overline{I}/I_t + \overline{I}I_t^{-1}\sigma_I(I)}{\sigma_Y(I)\,\rho + k\left(\sigma_Y(I)\,\overline{I}/I_t + \overline{I}I_t^{-1}\sigma_I(I)\right)}\,(w_i - \gamma_i) \\
&= \gamma_i + (\rho+k)\,\frac{\overline{I}/I_t\left[\sigma_Y(I) + \sigma_I(I)\right]}{\sigma_Y(I)\,\rho + k\overline{I}/I_t\left[\sigma_Y(I) + \sigma_I(I)\right]}\,(w_i - \gamma_i) \\
&= \gamma_i + (\rho+k)\,\frac{\overline{I}/I_t\sigma_M(I)}{\sigma_Y(I)\,\rho + k\overline{I}/I_t\sigma_M(I)}\,(w_i - \gamma_i)
\end{aligned}$$

where

$$\sigma_M(Y) = \sigma_Y(I) + \sigma_I(I)$$

Finally, substituting $\sigma_I(I) = \upsilon\sigma_Y(I)$ from definition (IA.5) and deleting $\sigma_Y(I)$ throughout, the result follows.

Similarly, we have that the amount in bonds is

$$
\begin{aligned}
B_{it} &= W_{it} - N_{it}P_t \\
&= Y_t\frac{1}{\rho}\left(\frac{\rho}{\rho+k}\gamma_i + \left(w_i - \frac{\rho}{\rho+k}\gamma_i\right)\overline{I}/I_t\right) - N_{it}Y_t\frac{\left(\rho+k\overline{I}/I_t\right)}{\rho\left(\rho+k\right)} \\
&= Y_t\frac{1}{\rho\left(\rho+k\right)}\left[\left(\rho\gamma_i + \left(w_i\left(\rho+k\right) - \rho\gamma_i\right)\overline{I}/I_t\right) - N_{it}\left(\rho+k\overline{I}/I_t\right)\right] \\
&= Y_t\frac{1}{\rho\left(\rho+k\right)}\left[\gamma_i\left(\rho+k\overline{I}/I_t\right) + w_i\left(\rho+k\right)\overline{I}/I_t - \gamma_i\left(\rho+k\right)\overline{I}/I_t - N_{it}\left(\rho+k\overline{I}/I_t\right)\right] \\
&= Y_t\frac{1}{\rho\left(\rho+k\right)}\left[\gamma_i\left(\rho+k\overline{I}/I_t\right) + \left(w_i-\gamma_i\right)\left(\rho+k\right)\overline{I}/I_t - N_{it}\left(\rho+k\overline{I}/I_t\right)\right] \\
&= Y_t\frac{1}{\rho}\left[\overline{I}/I_t - \frac{\overline{I}/I_t\sigma_M\left(I\right)}{\sigma_Y\left(I\right)\rho + k\overline{I}/I_t\sigma_M\left(I\right)}\left(\rho+k\overline{I}/I_t\right)\right]\left(w_i-\gamma_i\right) \\
&= Y_t\frac{1}{\rho}\left[\frac{\overline{I}/I_t\left[\sigma_Y\left(I\right)\rho + k\overline{I}/I_t\sigma_M\left(I\right)\right] - \overline{I}/I_t\sigma_M\left(I\right)\left(\rho+k\overline{I}/I_t\right)}{\sigma_Y\left(I\right)\rho + k\overline{I}/I_t\sigma_M\left(I\right)}\right]\left(w_i-\gamma_i\right) \\
&= -Y_t\left[\frac{\overline{I}/I_t\left(\sigma_M\left(I\right) - \sigma_Y\left(I\right)\right)}{\sigma_Y\left(I\right)\rho + k\overline{I}/I_t\sigma_M\left(I\right)}\right]\left(w_i-\gamma_i\right) \\
&= -Y_t\left[\frac{\overline{I}/I_t\left(\sigma_M\left(I\right)/\sigma_Y\left(I\right) - 1\right)}{\rho + k\overline{I}/I_t\sigma_M\left(I\right)/\sigma_Y\left(I\right)}\right]\left(w_i-\gamma_i\right)
\end{aligned}
$$

Finally, substituting $\sigma_I(I) = v\sigma_Y(I)$ from definition (IA.5) and deleting $\sigma_Y(I)$ throughout, the result follows. □

Proof of Corollary 1. Immediate from Proposition 2 and 3. The state price density and the price of stocks are independent of cross-sectional quantities. □

Proof of Corollary 2. We have

$$
\begin{aligned}
\frac{N_{it}P_t}{W_{it}} &= \frac{\sigma_{Wi}(I)}{\sigma_P(I)} \\
&= \frac{\sigma_Y(I) + \frac{\left(w_i - \frac{\rho}{\rho+k}\gamma_i\right)\bar{I}I_t^{-1}\sigma_I(I)}{\left(\frac{\rho}{\rho+k}\gamma_i + \left(w_i - \frac{\rho}{\rho+k}\gamma_i\right)\bar{I}/I_t\right)}}{\sigma_Y(I) + \frac{k\bar{I}I_t^{-1}\sigma_I(I)}{\left(\rho + k\bar{I}/I_t\right)}} \\
&= \frac{\sigma_Y(I) + \frac{(w_i(\rho+k)-\rho\gamma_i)\bar{I}I_t^{-1}\sigma_I(I)}{\left(\rho\gamma_i + (w_i(\rho+k)-\rho\gamma_i)\bar{I}/I_t\right)}}{\sigma_Y(I) + \frac{k\bar{I}I_t^{-1}\sigma_I(I)}{\left(\rho + k\bar{I}/I_t\right)}} \\
&= \frac{\sigma_Y(I) + \sigma_I(I)\left(\frac{(w_i(\rho+k)-\rho\gamma_i)\bar{I}I_t^{-1}}{\left(\rho\gamma_i + (w_i(\rho+k)-\rho\gamma_i)\bar{I}/I_t\right)}\right)}{\sigma_Y(I) + \sigma_I(I)\left(\frac{k\bar{I}I_t^{-1}}{\left(\rho + k\bar{I}/I_t\right)}\right)} \\
&= \frac{\sigma_Y(I) + \sigma_I(I)\left(1 - \frac{\rho}{\rho + [k + (\rho+k)(w_i-\gamma_i)/\gamma_i]\bar{I}/I_t}\right)}{\sigma_Y(I) + \sigma_I(I)\left(1 - \frac{\rho}{\left(\rho + k\bar{I}/I_t\right)}\right)}
\end{aligned}
$$

Finally, substituting $\sigma_I(I) = v\sigma_Y(I)$ from definition (IA.5) and deleting $\sigma_Y(I)$ throughout, the result follows. □

Derivation of expression (23)

Let

$$
\mathsf{SR}(I_t) \equiv \frac{E_t[dR_t - r_t dt]}{\sigma_P(I_t)} = (1+v)\,\sigma_Y(I_t) \qquad \text{and} \qquad \theta_i \equiv \frac{v\gamma_i}{(1+v)(\rho+k)}. \tag{IA.7}
$$

Finally define

$$
\Omega_i(I_t) \equiv \frac{\rho}{\rho + \left[k + \frac{(\rho+k)(\omega_i-\gamma_i)}{\gamma_i}\right]\bar{I}/I_t}
$$

The share of wealth invested in the risky security is

$$
\begin{aligned}
\frac{N_{i,t}P_t}{W_{i,t}} &= \left(\frac{\sigma_Y(I_t)}{\sigma_P(I_t)}\right)\left[1 + v\left(1 - \Omega_i(I_t)\right)\right] && \text{(IA.8)} \\
&= \left(\frac{\sigma_Y(I_t)}{\sigma_P(I_t)}\right)\left[1 + v\left(1 - \Omega_i(I_t)\right)\right] && \text{(IA.9)} \\
&= \left(\frac{(1+v)\,\sigma_Y(I_t)}{\sigma_P(I_t)}\right)\left[1 - \frac{v}{1+v}\Omega_i(I_t)\right] && \text{(IA.10)} \\
&= \frac{\mathsf{SR}(I_t)}{\sigma_P(I_t)}\left[1 - \frac{v}{1+v}\Omega_i(I_t)\right], && \text{(IA.11)}
\end{aligned}
$$

where we have made use of (IA.7).

The key is to show that

$$\begin{aligned}\Omega_i(I_t) &\equiv \frac{\rho}{\rho+\left[k+\frac{(\rho+k)(\omega_i-\gamma_i)}{\gamma_i}\right]\overline{I}/I_t} &\text{(IA.12)}\\ &= \frac{\rho\gamma_i}{\rho\gamma_i+\left[\rho(\omega_i-\gamma_i)+k\omega_i\right]\overline{I}/I_t} &\text{(IA.13)}\\ &= \left(\frac{Y_t}{\rho(\rho+k)}\right)\frac{\rho\gamma_i}{\frac{\left[\rho\gamma_i+\left[\rho(\omega_i-\gamma_i)+k\omega_i\right]\overline{I}/I_t\right]Y_t}{\rho(\rho+k)}} &\text{(IA.14)}\\ &= \left(\frac{\gamma_i}{\rho+k}\right)\frac{Y_t}{W_{i,t}} &\text{(IA.15)}\end{aligned}$$

Define θ_i as in (IA.7) and substitute in (IA.11) to obtain (23). □

Proof of Proposition ??. Part (a) follows immediately from the definitions. Part (b) is immediate from the expressions of debt-to-income and debt-to-wealth ratios. As for Part (c), debt-to-income ratio only depends on $H(I_t)$ which is decreasing in I_t. Therefore, debt-to-income ratio is procyclical. As for debt-to-wealth, we now show that

$$L_{it}/W_{it} = \frac{v\rho(\rho+k)(\omega_i-\gamma_i)H(I)}{\rho\gamma_i+((\rho+k)\omega_i-\gamma_i\rho)\overline{I}/I}$$

is increasing in I_t when I_t is below a (high) threshold I_i^*. It is useful to express the function as a function of $S = 1/I$:

$$\begin{aligned}L_{it}/W_{it} &= \frac{v\rho(\rho+k)(\omega_i-\gamma_i)H(I_t)}{\rho\gamma_i+((\rho+k)\omega_i-\gamma_i\rho)\overline{I}/I_t}\\ &= v\rho(\rho+k)(\omega_i-\gamma_i)\frac{\left(\frac{\overline{I}S_t}{\rho+k(1+v)\overline{I}S}\right)}{\left(\rho\gamma_i+((\rho+k)\omega_i-\gamma_i\rho)\overline{I}S_t\right)}\\ &= v(\omega_i-\gamma_i)\left(\frac{\overline{I}S_t\rho(\rho+k)}{\left(\rho+k(1+v)\overline{I}S_t\right)\left(\rho\gamma_i+((\rho+k)\omega_i-\gamma_i\rho)\overline{I}S_t\right)}\right)\\ &= v(\omega_i-\gamma_i)\left(\frac{\widetilde{S}_t\rho(\rho+k)}{\left(\rho+k(1+v)\widetilde{S}_t\right)\left(\rho\gamma_i+((\rho+k)\omega_i-\gamma_i\rho)\widetilde{S}_t\right)}\right)\end{aligned}$$

where $\widetilde{S}_t = \overline{I}S_t$. Recalling that leverage is positive for $\omega_i > \gamma_i$, we have then to show that

$$\begin{aligned}f\left(\widetilde{S}_t\right) &= \frac{\left(\rho+k(1+v)\widetilde{S}_t\right)\left(\rho\gamma_i+((\rho+k)\omega_i-\gamma_i\rho)\widetilde{S}_t\right)}{\widetilde{S}_t}\\ &= \frac{\left(\rho+k(1+v)\widetilde{S}_t\right)\rho\gamma_i+\left(\rho+k(1+v)\widetilde{S}_t\right)((\rho+k)\omega_i-\gamma_i\rho)\widetilde{S}_t}{\widetilde{S}_t}\\ &= \frac{a_i+b_i\widetilde{S}_t+c_i\widetilde{S}_t^2}{\widetilde{S}_t}\end{aligned}$$

is increasing in $\widetilde{S}_t$ where

$$\begin{aligned} a_i &= \rho^2 \gamma_i \\ b_i &= \rho \gamma_i k (1+v) + \rho((\rho+k)\omega_i - \gamma_i \rho) \\ c_i &= k(1+v)((\rho+k)\omega_i - \gamma_i \rho) \end{aligned}$$

Taking the first derivative, we have

$$\begin{aligned} f'\left(\widetilde{S}_t\right) &= \frac{\left[b_i + 2c_i \widetilde{S}_t\right]\widetilde{S}_t - \left[a_i + b_i \widetilde{S}_t + c_i \widetilde{S}_t^2\right]}{\widetilde{S}_t^2} = \frac{c_i \widetilde{S}_t^2 - a_i}{\widetilde{S}_t^2} \\ &= c_i - \frac{a_i}{\widetilde{S}_t^2} > 0 \end{aligned}$$

if and only if

$$\widetilde{S}_t > \sqrt{\frac{a_i}{c_i}} = \sqrt{\frac{\rho^2 \gamma_i}{k(1+v)((\rho+k)\omega_i - \gamma_i \rho)}}$$

Thus, the debt-to-wealth ratio L_{it}/W_{it} is increasing in I_t for

$$I_t < I_i^{**} = \overline{I}\frac{\sqrt{k(1+v)((\rho+k)\omega_i/\gamma_i - \rho)}}{\rho} \tag{IA.16}$$

This threshold is increasing in ω_i/γ_i. Moreover, recalling that $\omega_i/\gamma_i > 1$, we see that

$$I_i^{**} > I^{**} = \overline{I}\ \frac{k}{\rho}\ \sqrt{1+v} \tag{IA.17}$$

Hence, L_{it}/W_{it} is increasing in I_t for $I_t < I^{**}$. In the calibration we show that this threshold is in fact rarely reached by the process I_t. □

Proof of Corollary 3. The expressions of the drift and diffusion of dC_{it}/C_{it} stem from the application of Ito's lemma to the consumption $C_{it} = Y_t[\gamma_i + (\omega_i - \gamma_i)\overline{I}/I_t]$. The remaining part follows from the statement in the corollary. □

Proof of Corollary 4. Immediate from the fact $H(I_t)$ is decreasing and the fact that agents with $\omega_i - \gamma_i > 0$ are leveraged and have C_{it}/D_t that is decreasing in I_t. □

Proof of Proposition 5. Part (a) follows from the definition $\mathcal{D}_t$ in equation (30). Part (b) follow immediately from the definition of debt-to-output and debt-to-equity ratios. As for part (c), debt-to-output is procyclical as it only depends on $H(I)$, which is decreasing in I. As for debt-to-wealth, we now show that

$$\mathcal{D}/P = \frac{v\rho(\rho+k)K_1 H(I)}{\rho + k\overline{I}/I}$$

is increasing in I_t when I_t is below a (high) threshold I^*. It is useful to express the function as a function of $S = 1/I$:

$$\begin{aligned}
\mathcal{D}/P &= \frac{v\rho(\rho+k)K_1 H(I)}{\rho + k\overline{I}/I} \\
&= v\rho(\rho+k)K_1 \frac{\left(\frac{\overline{I}S}{\rho+k(1+v)\overline{I}S}\right)}{\left(\rho+k\overline{I}S\right)} \\
&= vK_1 \left(\frac{\overline{I}S\rho\left(\rho+k\right)}{\left(\rho+k\left(1+v\right)\overline{I}S\right)\left(\rho+k\overline{I}S\right)}\right) \\
&= vK_1 \left(\frac{\rho\left(\rho+k\right)}{\frac{\left(\rho+k\overline{I}S\right)^2}{\overline{I}S} + kv\left(\rho+k\overline{I}S\right)}\right)
\end{aligned}$$

We have to show that

$$\begin{aligned}
f(S) &= \frac{\left(\rho+k\overline{I}S\right)^2}{\overline{I}S} + kv\left(\rho+k\overline{I}S\right) \\
&= \frac{\left(\rho+k\overline{I}S\right)^2 + kv\left(\rho+k\overline{I}S\right)\overline{I}S}{\overline{I}S} \\
&= \frac{\rho^2 + (1+v)\left(k\overline{I}S\right)^2 + (2+v)\rho k\overline{I}S}{\overline{I}S} \\
&= \frac{\left[\rho^2 + (1+v)\left(k\widetilde{S}\right)^2 + (2+v)\rho k\widetilde{S}\right]}{\widetilde{S}}
\end{aligned}$$

is increasing in $\widetilde{S} = \overline{I}S$.

$$\begin{aligned}
f'\left(\widetilde{S}\right) &= \frac{\left[(1+v)2\left(k\widetilde{S}\right)k + (2+v)\rho k\right]\widetilde{S} - \left[\rho^2 + (1+v)\left(k\widetilde{S}\right)^2 + (2+v)\rho k\widetilde{S}\right]}{\widetilde{S}^2} \\
&= \frac{(1+v)2\left(k\widetilde{S}\right)^2 + (2+v)\rho k\widetilde{S} - \rho^2 - (1+v)\left(k\widetilde{S}\right)^2 - (2+v)\rho k\widetilde{S}}{\widetilde{S}^2} \\
&= \frac{(1+v)\left(k\widetilde{S}\right)^2 - \rho^2}{\widetilde{S}^2} > 0
\end{aligned}$$

if

$$\widetilde{S} > \frac{\rho}{k\sqrt{(1+v)}}$$

Thus, the debt-to-equity ratio $\mathcal{D}/P$ is increasing in I_t for $I_t < I^{**}$ where I^{**} is the same threshold in (IA.17). □

Proof of Corollary 5. The intermediary debt-to-output ratio is decreasing in I, i.e. in bad times, when the price P is also low and the volatility $\sigma_P(I)$ is high. The result follows. □

Proof of Corollary 6. Part (a) follows from the fact that from Proposition 5 the debt-to-output ratio $\mathcal{L}_t = \mathcal{D}/Y = f(I_t)$ is decreasing in I_t. This implies that its inverse $I_t = f^{-1}(\mathcal{L}_t) = q(\mathcal{L}_t)$ is also decreasing in $\mathcal{L}_t$. Thus, the market price of risk $\lambda^{\mathcal{D}/Y} = -q'(\mathcal{L})/q(\mathcal{L}) > 0$. Similarly, part (b) follows from the fact that again from Proposition 5 the ratio $\mathcal{L}_t = \mathcal{D}/P = g(I)$ is increasing in I_t for $I_t < I^*$ in equation (IA.17). Thus, in this range, $I_t = g^{-1}(\mathcal{L}_t) = q(\mathcal{L})$ is also increasing in $\mathcal{L}$ and thus in this case the market price of risk $\lambda^{\mathcal{D}/Y} = -q'(\mathcal{L})/q(\mathcal{L}) < 0$. □

Proof of Proposition 6. From market completeness, the first order condition of Lagrangean is

$$\frac{e^{-\rho t}}{C_{it} - \psi_{it} Y_t} = \frac{1}{\phi_i} M_t$$

We can write this as

$$\begin{aligned} C_{it} &= \psi_{it} Y_t + \phi_i M_t^{-1} e^{-\rho t} \\ &= \gamma_i \epsilon_{it} \left(1 - I_t^{-1}\right) Y_t + \phi_i M_t^{-1} e^{-\rho t} \end{aligned}$$

Aggregate and exploit the law of large numbers yields

$$\int \gamma_i \epsilon_{it} di = E^{CS}\left[\gamma_i \epsilon_{it}\right] = E^{CS}\left[\gamma_i\right] E^{CS}\left[\epsilon_{it}\right] = 1$$

and $\int \phi_i di = 1$. We thus obtain

$$\int C_{it} di = Y_t = \left(1 - I_t^{-1}\right) Y_t + M_t^{-1} e^{-\rho t}$$

Solving for M_t gives the same SDF as in the paper

$$M_t = e^{-\rho t} I_t Y_t^{-1}$$

The consumption share of agent i is thus

$$\begin{aligned} \frac{C_{it}}{Y_t} &= \gamma_i \epsilon_{it} \left(1 - I_t^{-1}\right) + \phi_i M_t^{-1} Y_t^{-1} e^{-\rho t} \\ &= \gamma_i \epsilon_{it} \left(1 - I_t^{-1}\right) + \phi_i I_t^{-1} \end{aligned}$$

In order to compute the Pareto weight ϕ_i we need to match agents' wealth to the initial endowment. From market completeness, the wealth of agent i at time t is given by

$$\begin{aligned} W_{it} &= M_t^{-1} E_t \left[\int_t^\infty M_s C_{is} ds\right] \\ &= M_t^{-1} E_t \left[\int_t^\infty e^{-\rho s} I_s Y_s^{-1} \left[\gamma_i \epsilon_{it} \left(1 - I_s^{-1}\right) + \phi_i I_s^{-1}\right] Y_s ds\right] \\ &= I_t^{-1} Y_t E_t \left[\int_t^\infty e^{-\rho(s-t)} \left[\gamma_i \epsilon_{it} \left(I_s - 1\right) + \phi_i\right] ds\right] \end{aligned}$$

From Fubini, and the fact that I_t and ϵ_{it} are independent, we obtain

$$W_{it} = I_t^{-1} Y_t \int_t^{\infty} e^{-\rho(s-t)} \left\{ \left(E_t \left[I_s \right] - 1 \right) \gamma_i E_t \left[\epsilon_{is} \right] + \phi_i \right\} ds$$

yielding

$$W_{it} = I_t^{-1} Y_t \int_t^{\infty} e^{-\rho(s-t)} \left\{ \left(\overline{I} + \left(I_t - \overline{I} \right) e^{-k(s-t)} - 1 \right) \gamma_i \left[1 + \left(\epsilon_{it} - 1 \right) e^{-\kappa(s-t)} \right] + \phi_i \right\} ds$$

Tedious algebra gives

$$W_{it} = Y_t \left\{ \frac{\left[\gamma_i \left(\overline{I} - 1 \right) + \phi_i \right]}{\rho} I_t^{-1} + \frac{\gamma_i \left(1 - \overline{I} I_t^{-1} \right)}{\rho + k} + \frac{\left(\overline{I} - 1 \right)}{\rho + \kappa} \gamma_i \left(\epsilon_{it} - 1 \right) I_t^{-1} + \frac{\left(1 - \overline{I} I_t^{-1} \right)}{\rho + k + \kappa} \gamma_i \left(\epsilon_{it} - 1 \right) \right\}$$

Assumption: Consider now time 0, assume we are at the stochastic steady state $I_0 = \overline{I}$ and $\epsilon_{i,0} = 1$, and $Y_0 = \rho$. We obtain

$$W_{i0} = \left[\gamma_i \left(\overline{I} - 1 \right) + \phi_i \right] \overline{I}^{-1}$$

Thus, the initial condition on wealth $W_{i0} = \omega_i$ gives the solution

$$\phi_i = \omega_i \overline{I} + \gamma_i \left(1 - \overline{I} \right)$$

Substitute back into consumption to obtain

$$\begin{aligned} \frac{C_{it}}{Y_t} &= \gamma_i \epsilon_{i,t} \left(1 - I_t^{-1} \right) + \phi_i I_t^{-1} \\ &= \gamma_i \epsilon_{i,t} \left(1 - I_t^{-1} \right) + \omega_i \overline{I} I_t^{-1} + \gamma_i \left(1 - \overline{I} \right) I_t^{-1} \end{aligned}$$

which proves point (a).

Substitute now ϕ_i also into the wealth of agent i to obtain

$$\begin{aligned} W_{it} &= Y_t \left\{ \frac{\left[\gamma_i \left(\overline{I} - 1 \right) + \phi_i \right]}{\rho} I_t^{-1} + \frac{\gamma_i \left(1 - \overline{I} I_t^{-1} \right)}{\rho + k} + \frac{\left(\overline{I} - 1 \right)}{\rho + \kappa} \gamma_i \left(\epsilon_{it} - 1 \right) I_t^{-1} + \frac{\left(1 - \overline{I} I_t^{-1} \right)}{\rho + k + \kappa} \gamma_i \left(\epsilon_{it} - 1 \right) \right\} \\ &= Y_t \left\{ \frac{\left[\gamma_i \left(\overline{I} - 1 \right) + \omega_i \overline{I} + \gamma_i \left(1 - \overline{I} \right) \right]}{\rho} I_t^{-1} + \frac{\gamma_i \left(1 - \overline{I} I_t^{-1} \right)}{\rho + k} + \frac{\left(\overline{I} - 1 \right)}{\rho + \kappa} \gamma_i \left(\epsilon_{it} - 1 \right) I_t^{-1} + \frac{\left(1 - \overline{I} I_t^{-1} \right)}{\rho + k + \kappa} \gamma_i \left(\epsilon_{it} - 1 \right) \right. \end{aligned}$$

We thus obtain

$$\frac{W_{it}}{Y_t} = \frac{\rho \gamma_i + \left(\omega_i \left(\rho + k \right) - \rho \gamma_i \right) \overline{I} I_t^{-1}}{\rho \left(\rho + k \right)} + \frac{\left(\rho + \kappa \right) + \left[\overline{I} k - \left(k + \rho + \kappa \right) \right] I_t^{-1}}{\left(\rho + \kappa \right) \left(\rho + k + \kappa \right)} \gamma_i \left(\epsilon_{it} - 1 \right)$$

which proves point (d).

We finally derive the position in stocks and bonds. Tedious algebra shows

$$\begin{aligned} dW_{it} &= o\left(dt \right) + \left[\begin{array}{c} \frac{\rho \gamma_i + \left(\omega_i \left(\rho + k \right) - \rho \gamma_i \right) \overline{I} S_t}{\rho \left(\rho + k \right)} + \frac{\left(\rho + \kappa \right) + \left[\overline{I} k - \left(k + \rho + \kappa \right) \right] S_t}{\left(\rho + \kappa \right) \left(\rho + k + \kappa \right)} \gamma_i \left(\epsilon_{it} - 1 \right) \\ + \frac{\left(\omega_i \left(\rho + k \right) - \rho \gamma_i \right) \overline{I} S_t v}{\rho \left(\rho + k \right)} + \frac{\left[\overline{I} k - \left(k + \rho + \kappa \right) \right]}{\left(\rho + \kappa \right) \left(\rho + k + \kappa \right)} \gamma_i \left(\epsilon_{it} - 1 \right) S_t v \end{array} \right] Y_t \sigma_Y \left(I_t \right) dZ_t \\ &\quad + Y_t \frac{\left(\rho + \kappa \right) + \left[\overline{I} k - \left(k + \rho + \kappa \right) \right] S_t}{\left(\rho + \kappa \right) \left(\rho + k + \kappa \right)} \gamma_i \sigma_\gamma \epsilon_{it} dZ_{it} \end{aligned}$$

In addition, the stock price dynamics is

$$dP_t = o\left(dt\right) + \left[\frac{\rho + k\overline{I}S_t}{\rho\left(\rho + k\right)} + v\frac{k\overline{I}S_t}{\rho\left(\rho + k\right)}\right] Y_t \sigma_Y\left(I_t\right) dZ_t$$

Wealth of agent i is $W_{it} = N_{it}P_t + B_{it} + N_{it}^F[F_{it}]$ where $F_{it} = 0$ is an instantaneously expiring futures to span the idiosyncratic shocks to individual preferences. We assume $dF_{it} = dZ_{it}$. Thus, from $dW_{it} = N_{it}dP_t + B_{it}r_t dt + N_{it}^F dZ_{it}$. It follows that

$$\begin{aligned}
N_{it} &= \frac{W_{it}\sigma_{W_{it}}}{P_t\sigma_P} = \frac{\sigma\left(dW_{it}\right)}{\sigma\left(dP\right)} \\
&= \frac{\rho\gamma_i + \left(\omega_i\left(\rho + k\right) - \rho\gamma_i\right)\left(1 + v\right)\overline{I}S_t}{\rho + k\left(1 + v\right)\overline{I}S_t} + \frac{\frac{(\rho+\kappa)+\left[\overline{I}k-(k+\rho+\kappa)\right](1+v)S_t}{(\rho+\kappa)(\rho+k+\kappa)}}{\frac{\rho+k(1+v)\overline{I}S_t}{\rho(\rho+k)}}\gamma_i\left(\epsilon_{it} - 1\right)
\end{aligned}$$

Tedious algebra yields

$$\begin{aligned}
N_{it} &= \gamma_i + \left(\omega_i - \gamma_i\right)\left(\rho + k\right)\left(1 + v\right) H\left(I_t\right) \\
&\quad + \frac{\rho\left(\rho + k\right)}{\left(\rho + \kappa\right)\left(\rho + k + \kappa\right)}\left[1 + \frac{\left[\kappa I_t - \left(k + \rho + \kappa\right)\left(1 + v\right)\right]}{\rho I_t + k\left(1 + v\right)\overline{I}}\right]\gamma_i\left(\epsilon_{it} - 1\right)
\end{aligned}$$

Define

$$H_2\left(I_t\right) = 1 + \frac{\left[\kappa I_t - \left(k + \rho + \kappa\right)\left(1 + v\right)\right]}{\rho I_t + k\left(1 + v\right)\overline{I}}$$

The result in part (c) follows from defining new $H_2(t)$ as

$$H_1(I_t) = \frac{\rho\left(\rho + k\right) H_2\left(I_t\right)}{\left(\rho + \kappa\right)\left(\rho + k + \kappa\right)} \tag{IA.18}$$

$$H_2\left(I_t\right) = 1 + \frac{\left[\left(k + \rho + \kappa\right)\left(1 + v\right) - \kappa I_t\right]}{\rho I_t + k\left(1 + v\right)\overline{I}} \tag{IA.19}$$

Finally, we compute the optimal bond position. The bond position is the residual from the stock position, assuming that the flow from the instantaneous futures accrue to the bonds. We then have

$$\begin{aligned}
B_{it} &= W_{it} - N_{it}P_t \\
&= Y_t\left\{\frac{\rho\gamma_i + \left(\omega_i\left(\rho + k\right) - \rho\gamma_i\right)\overline{I}I_t^{-1}}{\rho\left(\rho + k\right)} + \frac{\left(\rho + \kappa\right) + \left(\overline{I}k - \left(k + \rho + \kappa\right)\right)I_t^{-1}}{\left(\rho + \kappa\right)\left(\rho + k + \kappa\right)}\gamma_i\left(\epsilon_{it} - 1\right)\right\} \\
&\quad - Y_t\left(\frac{\rho + k\overline{I}I_t^{-1}}{\rho\left(\rho + k\right)}\right)\left\{\gamma_i + \left(\omega_i - \gamma_i\right)\left(\rho + k\right)\left(1 + v\right)H\left(I_t\right) + \frac{\rho\left(\rho + k\right)}{\left(\rho + \kappa\right)\left(\rho + k + \kappa\right)}H_2\left(I_t\right)\gamma_i\left(\epsilon_{it} - 1\right)\right\}
\end{aligned}$$

Much algebra gives

$$\begin{aligned}
\frac{B_{it}}{Y_t} &= \left[\overline{I}I_t^{-1} - \left(\rho + k\overline{I}I_t^{-1}\right)\left(1 + v\right)\frac{\overline{I}}{\rho I_t + k\left(1 + v\right)\overline{I}}\right]\frac{1}{\rho}\left(\omega_i - \gamma_i\right) \\
&\quad + \frac{\left(\rho + \kappa\right) + \left(\overline{I}k - \left(k + \rho + \kappa\right)\right)I_t^{-1}}{\left(\rho + \kappa\right)\left(\rho + k + \kappa\right)}\gamma_i\left(\epsilon_{it} - 1\right) - \left(\frac{\rho + k\overline{I}I_t^{-1}}{\rho\left(\rho + k\right)}\right)\frac{\rho\left(\rho + k\right)}{\left(\rho + \kappa\right)\left(\rho + k + \kappa\right)}H_2\left(I_t\right)\gamma_i\left(\epsilon_{it} -\right.
\end{aligned}$$

The first line is the same as in the proof of B_{it}/Y_t for the case without idiosyncratic shocks, and thus we have

$$\begin{aligned}\frac{B_{it}}{Y_t} &= -\left(\omega_i-\gamma_i\right)vH\left(I_t\right)\\ &+\left[\frac{(\rho+\kappa)+\left(\bar{I}k-(k+\rho+\kappa)\right)I_t^{-1}}{(\rho+\kappa)(\rho+k+\kappa)}-\left(\frac{\rho+k\bar{I}I_t^{-1}}{\rho(\rho+k)}\right)\frac{\rho(\rho+k)}{(\rho+\kappa)(\rho+k+\kappa)}H_2\left(I_t\right)\right]\gamma_i\left(\epsilon_{it}-1\right)\end{aligned}$$

Substitute now $\mathrm{H}_2(I)$ and further algebra gives

$$\begin{aligned}\frac{B_{it}}{Y_t} &= -\left(\omega_i-\gamma_i\right)(\rho+k)\,vH\left(I_t\right)\\ &+\frac{v}{(\rho+\kappa)(\rho+k+\kappa)}\left[\frac{\kappa k\bar{I}+\rho(k+\rho+\kappa)}{\rho I_t+k(1+v)\bar{I}}\right]\gamma_i\left(\epsilon_{it}-1\right)\end{aligned}$$

or

$$\frac{B_{it}}{Y_t}=\left[-\left(\omega_i-\gamma_i\right)(\rho+k)+\frac{\kappa k+\rho\bar{I}^{-1}(k+\rho+\kappa)}{(\rho+\kappa)(\rho+k+\kappa)}\gamma_i\left(\epsilon_{it}-1\right)\right]vH\left(I_t\right)$$

Defining

$$Q=\left(\frac{\kappa k+\rho\bar{I}^{-1}(k+\rho+\kappa)}{(\rho+\kappa)(\rho+k+\kappa)}\right) \tag{IA.20}$$

gives point (b). Q.E.D.

IA2. Estimating household total and systematic consumption volatility

A challenge in the literature regarding the estimation of consumption volatility – the systematic and idiosyncratic components – is the lack of reliable high frequency panel data. In this Appendix we illustrate how we can use only cross-sectional information across households and then the time series across cohorts to estimate both components.

Consider the simple continuous time model, which generalizes the one derived in the model as we allow consumption to have cross-sectionally independent shocks:

$$\frac{dC_{it}}{C_{it}}=\mu_{it}dt+\sigma_{it}dZ_{it} \tag{IA.21}$$

In this process, both μ_{it} and σ_{it} are cross-sectionally different from each other and time varying. We are interested in estimating σ_{it}. From Ito's lemma we have

$$d\log\left(C_{it}\right)=\left(\mu_{it}-\frac{1}{2}\sigma_{it}^2\right)dt+\sigma_{it}dZ_{it} \tag{IA.22}$$

Therefore, for every i and t, we have

$$\widehat{\sigma}_{it}^2 = \frac{2}{dt}\left[\frac{dC_{it}}{C_{it}} - d\log\left(C_{it}\right)\right] \tag{IA.23}$$

This quantity is independent of dZ_{it} (it is a dt term) and on whether shocks are correlated with each other or not. Therefore, the (rescaled) difference between arithmetic and log consumption growth isolates the consumption variance of agent i at time t. This is a (noisy) observation of variance itself, and we are going to treat it as such.

In our model all consumption processes are perfectly correlated, and there are no idiosyncratic shocks. To calibrate the model we thus assume a common shock to dZ_{it}, that is

$$dZ_{it} = \rho dZ_t + \sqrt{1-\rho^2}dZ_{it}^*$$

where dZ_{it}^* are uncorrelated across i. This assumption implies that all consumption process across every two agents have correlation ρ^2:

$$Corr\left(\frac{dC_{it}}{C_{it}}\frac{dC_{jt}}{C_{jt}}\right) = \rho^2 dt$$

This assumption can be relaxed but given the scope of the current calibration it suffices for our purposes.

Consider now the cross-sectional average of consumption growth $E_t^{CS}\left[\frac{dC_{it}}{C_{it}}\right]$. This quantity follows the dynamic process

$$\begin{aligned} E_t^{CS}\left[\frac{dC_{it}}{C_{it}}\right] &= E_t^{CS}\left[g_{it}\right]dt + E_t^{CS}\left[\sigma_{it}dZ_{it}\right] \\ &= E_t^{CS}\left[g_{it}\right]dt + \rho E_t^{CS}\left[\sigma_{it}\right]dZ_t + \sqrt{(1-\rho^2)}E_t^{CS}\left[\sigma_{it}dZ_{it}^*\right] \end{aligned}$$

From the law of large numbers the idiosyncratic shocks average to zero

$$E_t^{CS}\left[\sigma_{it}dZ_{it}^*\right] = E_t^{CS}\left[\sigma_{it}\right]E^{CS}\left[dZ_{it}^*\right] = 0$$

Therefore the average arithmetic consumption growth follows

$$E_t^{CS}\left[\frac{dC_{it}}{C_{it}}\right] = E_t^{CS}\left[g_{it}\right]dt + E_t^{CS}\left[\sigma_{it}\right]\rho dZ_t$$

Hence, the squared variation of average consumption growth in continuous time has

$$\left(E_t^{CS}\left[\frac{dC_{it}}{C_{it}}\right]\right)^2 = E_t^{CS}\left[\sigma_{it}\right]^2\rho^2 dt + o\left(dt\right)$$

That is, we can identify the average systematic volatility of consumption growth from the squared variation of the cross-sectional average of consumption growth, a result that is not surprising.

We are interested however to also identify the whole distribution of systematic volatility $\{\sigma_{it}^2\rho^2\}_i$. Given our estimates of σ_{it}^2 obtained earlier we just need to estimate ρ^2, which can be done from the following estimator:

$$\widehat{\rho}^2 = \frac{E_t^{CS}\left[\frac{dC_{it}}{C_{it}}\right]^2}{E_t^{CS}\left[\sigma_{it}\right]^2}/dt \tag{IA.24}$$

The systematic variance of agent i at time t is then

$$V_{it}^2 = \widehat{\sigma}_{it}^2\widehat{\rho}^2 \tag{IA.25}$$

To conclude this section, we note that to estimate time t quantities – idiosyncratic and total volatility components – we only need cross-sectional information. We then use time series information across cohorts of households to compute averages.

CEX Data

We exploit the dataset of Kocherlakota and Pistaferri (2009) and Toda and Welsh (2015). We refer the reader to those papers for a more detailed description of the data. In a nutshell, the data are from the survey of consumer expenditure (CEX). Households are surveyed for four consecutive quarters, in January, February, and March cycles. Thus, the growth rate can be observed at most at quarterly frequency, i.e. $dt = 0.25$. While this is a large time difference, Monte Carlo simulations indicate that the methodology above provides reliable estimates for the distribution of consumption volatility.

For each agent i, we have 3 observations of its variance as in (IA.23). To minimize the impact of seasonalities, we then take the average of the three observations of $\widehat{\sigma}_{it}^2$ across the three quarters

$$\widehat{V}_t^i = \frac{1}{3}\widehat{\sigma}_{it}^2 + \frac{1}{3}\widehat{\sigma}_{it+.25}^2 + \frac{1}{3}\widehat{\sigma}_{it+.5}^2$$

For every year t in a given cycle (Jan, Feb, and Mar), we can then compute the distribution of consumption volatility across households. For instance, we can compute the mean, the median, and various percentiles α

$$\widehat{V}_t^{Ave} = Average\left[\widehat{V}_t^i\right]; \quad \widehat{V}_t^{Med} = Median\left[V_t^i\right]; \quad \widehat{V}_t^{\alpha} = Percentile\left[V_t^i, \alpha\right]$$

Similarly, for every t we can compute an observation for $\widehat{\rho}_t$ from estimator (IA.24). We can thus obtain the systematic component of volatilities as above:

$$\widehat{V}_t^{Sys,Ave} = Average\left[\widehat{V}_t^i\widehat{\rho}_t^2\right]; \quad \widehat{V}_t^{Sys,Med} = Median\left[V_t^i\widehat{\rho}_t^2\right]; \quad \widehat{V}_t^{Sys,\alpha} = Percentile\left[V_t^i\widehat{\rho}_t^2, \alpha\right]$$

We finally take average across cohorts (Jan, Feb, and March), and finally across time. Panel A of Table 2 contains the results.

Table IA.2: Cross-Sectional Parameters and Household Consumption Moments. This table reports the distribution of household consumption growth and their quarterly systematic volatility in artificial data. We simulate the model with 200,000 households for a period of 1,000 years. As in the data, estimates are performed on quarterly data. The parameters of the model are as in Table A.1, except for the preference and Pareto weight parameters γ_i and ϕ_i, which are reported in the first two columns. We assume $\gamma_i \sim Uniform[\underline{\gamma}, \overline{\gamma}]$ and $\phi_i \sim$ Log Normal$(-\frac{1}{2}\sigma_\phi^2, \sigma_\phi^2)$. Each row corresponds to a different parameter choice.

Households Quarterly Consumption Moments. Model

$U[\underline{\gamma}, \overline{\gamma}]$	σ_ϕ	Arithmetic Growth Rate (%)			Volatility (%)		
		Mean	Median	Std. Dev.	Mean	Median	Std. Dev.
$U[0,2]$	1.00	1.11	0.53	7.94	10.95	6.72	14.41
$U[0,2]$	0.75	0.89	0.51	6.30	8.74	5.66	11.39
$\mathbf{U[0,2]}$	**0.50**	**0.75**	**0.51**	**4.80**	**7.00**	**4.60**	**8.78**
$U[0,2]$	0.25	0.70	0.51	3.92	6.20	3.81	7.72
$U[0,2]$	0.00	0.69	0.52	3.63	6.00	3.52	7.67
$U[0,2]$	$\omega_i = 1$	0.81	0.52	5.21	7.70	5.67	8.66
$U[1,1]$	1.00	1.05	0.58	7.13	10.42	5.92	13.81
$U[1,1]$	0.75	0.76	0.52	4.94	7.06	4.35	9.44
$U[1,1]$	0.50	0.60	0.51	3.04	4.34	2.81	5.57
$U[1,1]$	0.25	0.53	0.51	1.43	2.23	1.62	2.61
$U[1,1]$	0.00	0.52	0.52	0.00	1.27	1.27	0.00

IA3. Cross-Sectional Heterogeneity

In this section we illustrate the model's implications for alternative parameterizations for the cross-sectional parameters to highlight the importance of differences in endowments and difference in utility functions. Table IA.2 provides the distribution of consumption growth and volatility from the model for different parametrizations, which should be compared to the estimates in Panel A of Table 2 in the paper. Next two sections shows the implications for the distribution of leverage for two specific cases in Table IA.2: Homogeneous preferences ($\overline{\gamma} = \underline{\gamma} = 1$) and homogeneous endowments ($\omega_i = 1$)

IA3.1. Homogeneous Preferences

We first investigate the impact of homogeneous preferences on our results. Recall that in our model agents' endowment heterogeneity still generates cross-sectional differences in risk aversion, and in agents' leverage and portfolio holdings. From Panel B of Table IA.2 we see that moving from heterogeneous to homogeneous preferences yields a large decrease in households' dispersion in consumption growth (from 4.8% to 3.04%) and in household quarterly consumption volatility (from 8.78% to 5.57%). That is, endowment heterogeneity still generates differential systematic volatility of household consumption but not as large as the data seem to indicate (Panel A).

In addition, Panel A of Figure IA.1 shows that homogeneous preferences with endowment

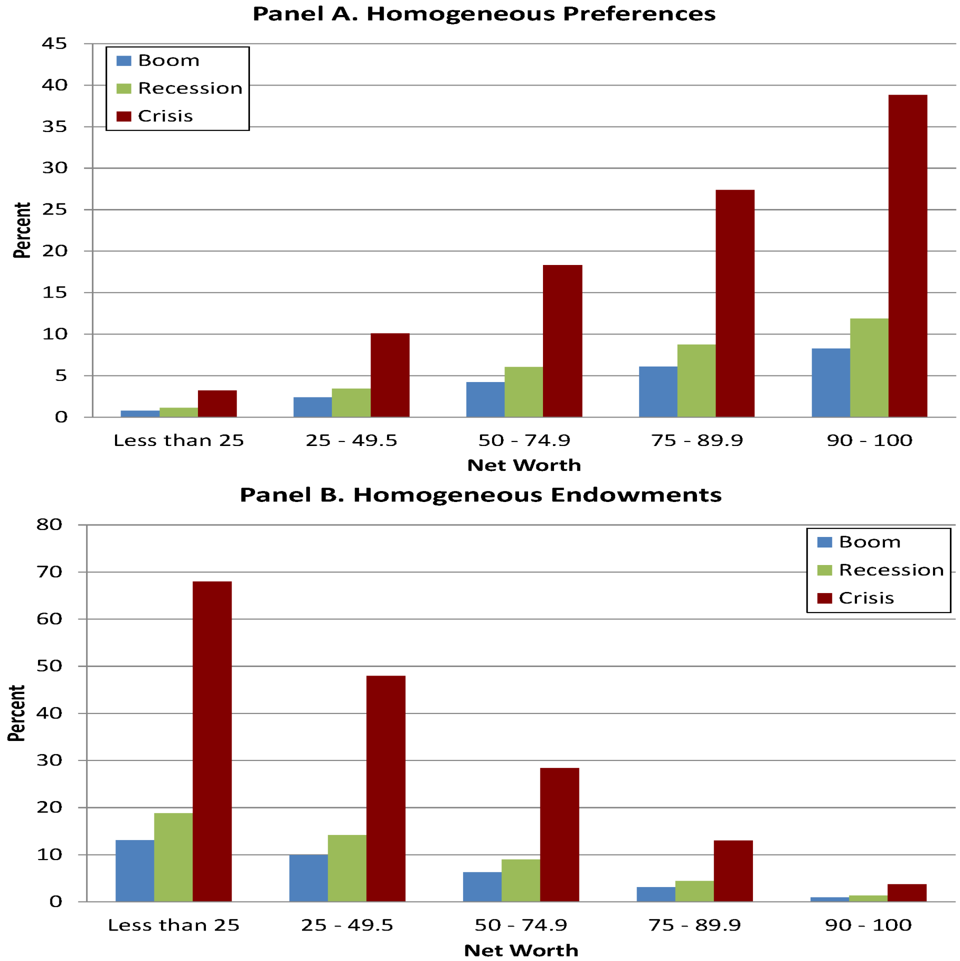


Figure IA.1: Debt-to-assets ratios across the wealth distribution. Both panels plot the distribution of debt-to-asset ratios of agents who take on debt in simulations during three types of periods: Booms (S_t high), recessions (S_t low) and crisis (S_t very low). Panel A reports the case in which all agents have homogeneous preferences $\gamma_i = 1$ for all i. Panel B reports the case in which all agents have identical endowments $\omega_i = 1$ for all i.

heterogeneity produces a debt/asset ratio that is counterfactually *increasing* in agents' net worth (see Figure 5). That is, in this case, agents with higher endowment are less risk averse and thus take on leverage.

IA3.2. Homogeneous Endowments

The second interesting case is one in which agents have still heterogeneous preferences $\gamma_i \sim U[0, 2]$ but have homogeneous endowments $\omega_i = 1$. In this case, Pareto weights are given by expression (10). Panel B of Table IA.2, line $\omega_i = 1$, shows that the implications about consumption behavior are very similar to those with larger heterogeneity in Pareto weights. Indeed, Panel B of Figure IA.1 shows a similar leverage pattern across net worth as in Panel A of Figure 5.

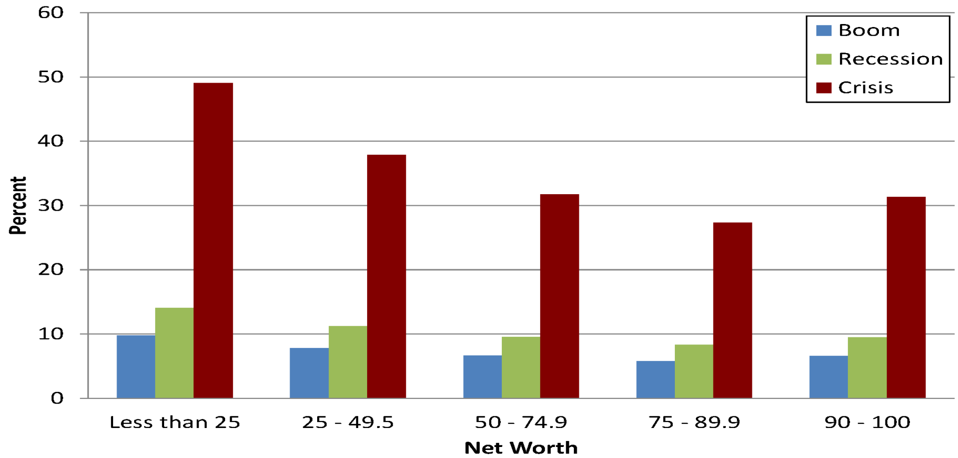


Figure IA.2: Debt-to-assets ratios across the wealth distribution for higher dispersion of endowments. This figure plots the distribution of debt-to-asset ratios of agents who take on debt in simulations during three types of periods: Booms (S_t high), recessions (S_t low) and crisis (S_t very low). Parameters are as in Figure 5 in the paper except that $\sigma_w = 1$.

IA3.3. A U-Shape in Leverage

The homegeneous preferences and homogeneous endowments discussed in the previous two sections highlight two dramatically different pattern in leverage (debt/assets) versus net-worth of the leveraged agents in the economy. The main body uses a calibration for the cross-section of agents that matches the volatility of systematic consumption as in the data, and it is consistent with a declining pattern in leverage versus net worth. We consider here the pattern that realizes when we choose $\sigma_w = 1$ instead of $\sigma_w = 0.5$. As shown in Panel B of Table IA.2 this choice generates too high average systematic volatility. Figure IA.2 plots the debt-to-asset distribution across net-worth bins and shows a mildly U-shape distribution of leverage: Agents with low net-worth are the most leveraged, the ones with the 75-90 percentile are the least leveraged and those in the top decile (90-100) have higher leverage than the those in the 75-90 percentile.

IA3.4. Leverage and Portfolio Allocation

Figure 5 in the body of the paper shows that the model implies that poorer households borrow more.[1] In that figure, we sort agents on their net worth at the time in crisis, as it is done in Figure 2 in the data, as this procedure emphasizes the impact of crisis on the wealth distribution itself. However, the decreasing pattern is visible, albeit less strongly, also if we sort agents period by period, as shown in Figure IA.3.

Figure IA.4 shows the portfolio allocation all of households under the three scenarios

[1] The households plotted in Figure 5 and in Figure IA.3 are only households who have debt, as in Figure 2 in the data.

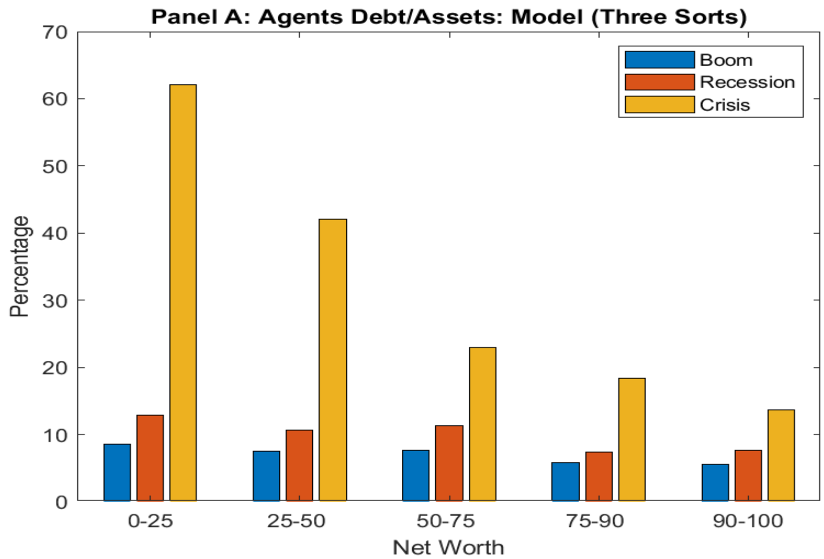


Figure IA.3: Debt-to-assets ratios across the wealth distribution in the model . This figure plots the distribution of debt-to-asset ratios of households who take on debt in simulations during three types of periods: Booms (I_t low), recessions (I_t high) and crisis (I_t very high). Households are sorted by the net worth on each scenario (boom, recession, crisis).

boom, recession and crisis. As it can be seen, during booms and recession, the portfolio holdings are mildly U shaped, with richer agents holding more in risky asset than poorer agents. However, during crisis times, the position of the risky asset of poor households increases while the one of rich households decreases, generating a declining pattern. Indeed, poor agents borrow to purchase the risky asset. As discussed in the paper, in our simple model they do so to purchase intermediary equity, while in reality such agents borrow to invest in housing. However, even if such leveraged agents actively deleverage (the amount of declines), they find themselves with more than 100% in the risky asset as their net worth is highly leveraged due to the loss in value of equity holdings. Note that from Proposition 2, however, conditioning on γ_i, higher wealth W_{it} correlates with larger position in stocks cross-sectionally (due to higher endowment ω_i). On the time series dimension, however, it depends it does so if only if $\omega_i > \gamma_i$. In other words, during bad times (high I_t), agents with $\omega_i > \gamma_i$ decrease their position in stocks while the others increase it. This is a market clearing condition, which highlights that even if risk aversion of households increase in bad times, some of them will need to increase the position in the risky assets, by market clearing.

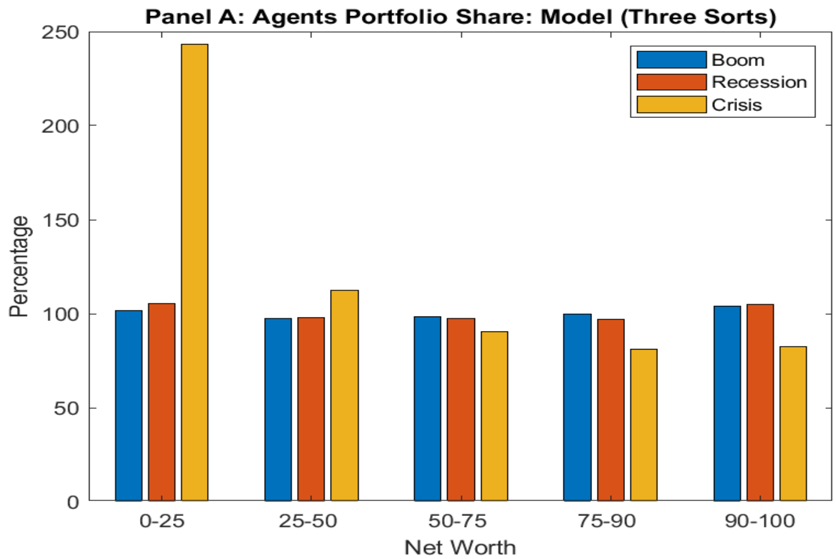


Figure IA.4: Portfolio share in risky asset across the wealth distribution in the model . This figure plots the distribution of the share of portfolio holdings in the risky asset, $N_{it}P_t/W_{it}$, of all households in simulations during three types of periods: Booms (I_t low), recessions (I_t high) and crisis (I_t very high). Households are sorted by the net worth on each scenario (boom, recession, crisis).

IA4. Household Leverage

IA4.1. Equity and Private Business as Leveraged Investments

In this section we use the CSF data to include the effective leverage of households implied by their investments in equity and private businesses.[2] To begin and get a better idea of how important could be the indirect leverage from investment in stocks, private equity, etc Figures IA.5 and IA.6 show some graphs depicting households' asset composition by percentiles. As we would expect the proportion of financial assets is increasing in net worth. In these graphs we also include "business interests" which is classified as a non-financial asset and it corresponds to businesses owned by the household (not publicly traded). We can see that those in the top of the net worth distribution have a large proportion of their assets in business interests (up to 40% in the top percentile). Similarly, the share of assets in stocks, mutual funds, and retirement account are increasing in net worth.

Given the evidence above, we now adjust household leverage for the leverage that is already implicit in their investments. Based on the decomposition of assets, Equity and Business Interests (BUS), there are the two categories where we could include adjustments. The other category that includes equity is Other Financial Assets (OTHFIN). OTHFIN includes Non-Public Stocks. However, OTHFIN as share of total assets is very small and if we decompose OTHFIN even further, we find that in 2007 no household report holdings of Non-Public Stocks and in 2010 only 5 households report holdings. We thus do nothing to

[2]We thank Alejandro Hojo Suarez for excellent research assistance.

Figure IA.5: Financial and non-financial asset by net-worth percentile

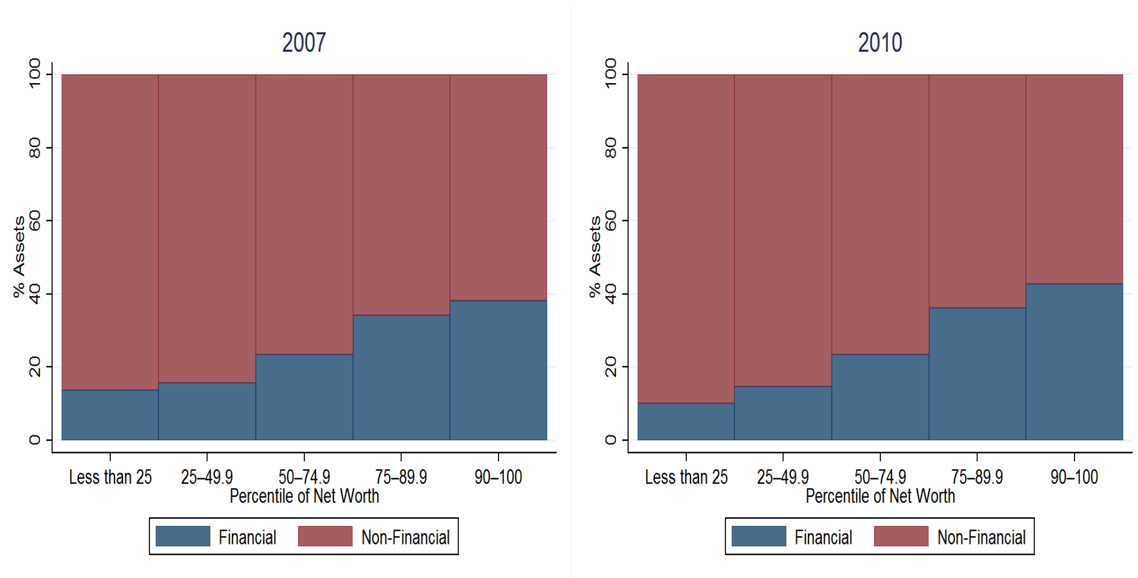
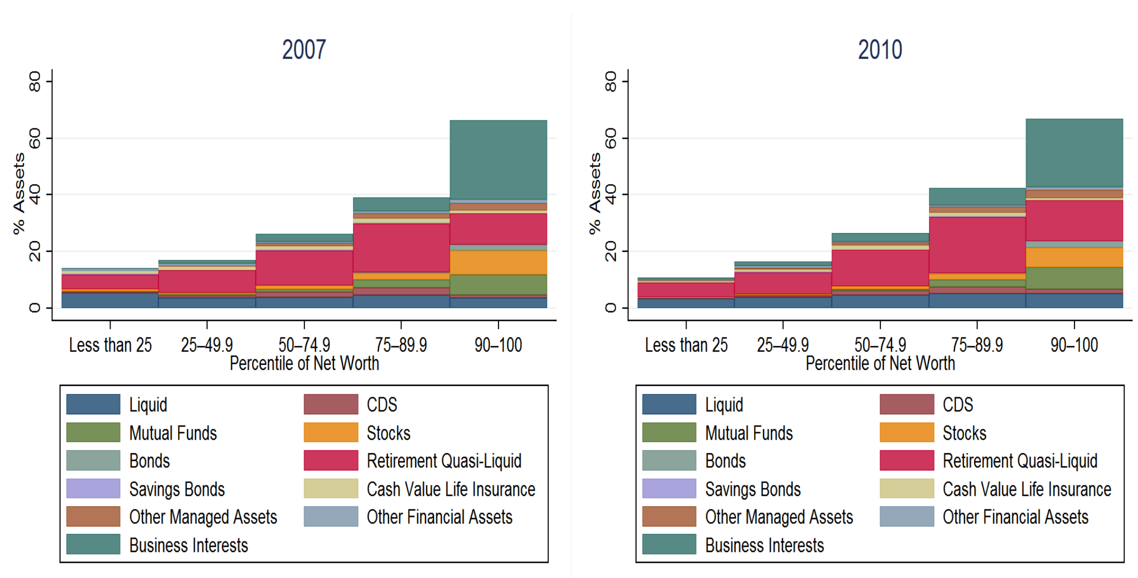


Figure IA.6: Asset composition by net-worth percentile



OTHFIN.

We make three adjustments to take into account implicit leverage. In general, to adjust debt and assets by indirect leverage we need to keep in mind that the following accounting equation must hold at all times:

$$\text{Net Worth} = \text{Assets - Debt}$$

We can decompose Assets as:

$$\text{Assets} = \text{Equity} + \text{BUS} + \text{Other Assets}$$

Now, Equity reported by households must be equal to shareholders equity from the firms' balance sheet

$$\text{Equity} = \text{Firms Assets - Firm Debt}$$

We consider two cases for the ratio $\frac{\text{Firm Debt}}{\text{Firm Assets}}$, namely, 20% and 30% (see Graham, Leary, and Roberts (2015), Table 1, Panel C and Figure 2, Panel A). For instance, if we consider a 20% Firm Debt/Firm Asset ratio, the calculations are as follows:

$$\text{Net Worth } = \underbrace{\frac{1}{0.8}\text{Equity } + \text{BUS} + \text{OtherAssets}}_{\text{Adj1.Assets}} - \underbrace{\left(\text{Debt} + \frac{0.2}{0.8}\text{Equity}\right)}_{\text{Adj1.Debt}}$$

IA4.1.1. First Adjustment

Assuming a 20% debt/asset ratio, the first adjustment is thus:

$$\begin{aligned}\text{Adj1.Assets} &= \text{Assets} + \frac{0.2}{0.8}\text{ Equity}\\ \text{Adj1.Debt} &= \text{Debt} + \frac{0.2}{0.8}\text{ Equity}\end{aligned}$$

Columns 4 and 5 of Table IA.3 shows the leverage ratio in 2007 and 2009, respectively, after the first adjustment. Panel A reports the case in which the implicit equity leverage is 20% while Panel B reports the case with implicit leverage at 30%. Compared to Columns 2 and 3 (no adjustments) in the respective panels, the implicit leverage of equity indeed increases the effective leverage of households in the top percentile of the net-worth distribution, but mildly. The decreasing pattern of leverage in net worth is unchanged.

IA4.1.2. Second Adjustment

The second adjustment is related to business interests. Business interests is constructed as net equity if businesses were sold today (NPrEq), plus loans from household to business

Table IA.3: Household Leverage: Robustness. This table reports households leverage from SCF data with adjustments for the leverage implicit in the assets in their portfolios. We consider three adjustments. Adjustment 1 adjusts debt and assets for the leverage implicit in equity. Adjustment 2 adjusts debt and assets for the leverage implicit in equity and the collateralized debt in private businesses (PB). Adjustment 3 adjusts debt and assets for the leverage implicit in equity and in private businesses. The implicit leverage in equity is in the titles of the two panels.

Panel A. 20% Implicit Leverage in Equity

Percentile of Net Worth	No Adjustments		Adjustment 1		Adjustment 2		Adjustment 3 (30% leverage in PB)		Adjustment 3 (60% leverage in PB)	
in 2009	2007	2009	2007	2009	2007	2009	2007	2009	2007	2009
Less 25	74.8	134.6	74.7	134.2	74.8	133.6	75.0	133.2	73.7	131.4
25 - 49.9	50.6	65.2	50.0	64.5	50.5	64.6	51.5	64.7	51.9	62.9
50-74.9	32.0	40.5	31.8	39.2	32.5	39.4	33.6	39.9	34.3	38.4
75-89.9	20.7	24.4	21.1	23.2	24.3	23.5	25.7	24.8	29.1	27.3
90-100	7.8	9.7	10.4	10.7	14.2	15.1	21.0	21.4	35.6	35.1

Panel B. 30% Implicit Leverage in Equity

Percentile of Net Worth	No Adjustments		Adjustment 1		Adjustment 2		Adjustment 3 (30% leverage in PB)		Adjustment 3 (60% leverage in PB)	
in 2009	2007	2009	2007	2009	2007	2009	2007	2009	2007	2009
Less 25	74.8	134.6	74.9	134.0	74.9	133.4	75.1	133.0	73.9	131.2
25 - 49.9	50.6	65.2	50.5	64.9	51.0	65.0	52.0	65.1	52.4	63.3
50-74.9	32.0	40.5	33.2	40.2	33.9	40.4	34.9	40.9	35.6	39.4
75-89.9	20.7	24.4	23.5	25.1	26.5	25.4	27.8	26.6	30.9	28.9
90-100	7.8	9.7	13.6	13.4	17.1	17.5	23.4	23.4	37.2	36.4

(L), minus loans from business to household not previously reported (BLoan), plus value of personal assets used as collateral for business loans that were reported earlier (COL_Rep).

$$\text{BUS} = \text{NPrEq} + \text{L} - \text{BLoan} + \text{COL\_Rep}$$

We can adjust BUS by adding back the loans from the business to the household not previously reported (BLoan) and add the value of personal assets used as collateral for business loans that were NOT reported earlier (COL_NORep).

$$\text{Adj.BUS} = \text{BUS} + \text{BLoan} + \text{COL\_NORep}$$

Still assuming an implicit debt/asset = 20% for illustration, the Net Worth is then:

$$\text{Net Worth} = \underbrace{\frac{1}{0.8}\text{Equity} + \text{Adj.BUS} + \text{OtherAssets}}_{\text{Adj2.Assets}} - \underbrace{\left(\text{Debt} + \frac{0.2}{0.8}\text{Equity} + \text{BLoan} + \text{COL\_NORep}\right)}_{\text{Adj2.Debt}}$$

The second adjustment is

$$\begin{aligned} \text{Adj2.Assets} &= \text{Assets} + \frac{0.2}{0.8}\text{Equity} + \text{BLoan} + \text{COL\_NORep} \\ \text{Adj2.Debt} &= \text{Debt} + \frac{0.2}{0.8}\text{Equity} + \text{BLoan} + \text{COL\_NORep} \end{aligned}$$

Columns 6 and 7 of Table IA.3 shows the leverage ratio in 2007 and 2009, respectively, after the first and second adjustment. Again, Panel A reports the case in which the implicit equity leverage is 20% while Panel B reports the case with implicit leverage at 30%. Once again, compared to the previous cases in the respective panels, adding the business loans collateralized by personal assets increases the effective leverage of households in the top percentiles of the net-worth distributions, but again mildly. In particular, the decreasing pattern of leverage in net worth is unchanged.

IA4.1.3. Third adjustment

The second adjustment only considered the implicit leverage in private businessed that is backed with personal assets. We include this third adjustment to take into account that most of the leverage in private business is not necessarily backed with personal assets. In a similar spirit as the first adjustment, we assume a leverage ratio for private business of 30% and 60%. This latter number is mostly to show the impact of such high leverage on the calculations, but it is not representative of the average leverage of private businesses, which is around 20%-30% (see Table II, Cole (2013).) Still assuming a 20% implicit equity leverage and a 30% implicit leverage for private businesses, the adjustment is then

$$\begin{aligned} \text{Adj3.Assets} &= \max\left\{\text{Assets} + \frac{0.2}{0.8}\text{Equity} + \text{BLoan} + \frac{0.3}{0.7}\text{NPrEq}\ ,\ \text{Adj2.Assets}\right\} \\ \text{Adj3.Debt} &= \max\left\{\text{Debt} + \frac{0.2}{0.8}\text{Equity} + \text{BLoan} + \frac{0.3}{0.7}\text{NPrEq}\ ,\ \text{Adj2.Debt}\right\} \end{aligned}$$

Columns 8 and 9 of Table IA.3 shows the leverage ratio in 2007 and 2009, respectively, after the three adjustments, with a 30% implicit leverage in private businesses. As before, Panel A reports the case in which the implicit equity leverage is 20% while Panel B reports the case with implicit equity leverage at 30%. Once again, compared to the previous cases in the respective panels, adding the implicit leverage of private businesses increases the effective leverage of households in the top percentiles of the net-worth distributions. Still, the decreasing pattern of leverage in net worth is unchanged. To obtain a mildly U-shaped pattern in households' leverage with respect to net-worth one has to push the implicit leverage of private business much higher. For instance, the last two columns shows the case with 60% implied private business leverage and indeed we now see that households in the top 10% of net-worth distribution have more leverage than those in the 75-90% group. However, 60% average leverage of private businesses is a very high number compared to available estimates (see Cole (2013, Table II)).

IA4.2. Household Leverage during the Spanish Crisis

In this section we report some additional evidence on household leverage around crises by using survey data from Spain.[3] For several years now the Bank of Spain has undertaken an official survey on Spanish households to gauge their financial position. The survey is known as the Encuesta Financiera de las Familias or EFF. This survey is similar to the Survey of Consumer Finances in U.S.[4] and it is designed to give as complete a view as possible of the financial position of Spanish households.[5] There are several waves available (2002, 2005, 2008, 2011, and 2014), which we can use to observe the financial position of Spanish households during the years in which the Spanish economy grew at a robust pace and the crisis years that start in 2008. A subtle issue concerns the moment the households are observed; the "wave year" stradles, typically, the previous year, when part of the fieldwork was conducted, and the year of the wave itself, when the fieldwork was completed. For instance for the 2008 survey the fieldwork lasted for eight months, between November 2008 and July 2009 and half of the surveys had already been conducted by March 2009 (see Bover, 2011). Instead in the case of the 2014 survey the fieldwork took place between September 2014 and March 2015 (see Bank of Spain, 2017).

With all these caveats in mind, Figure IA.7 shows the pattern of leverage once households are sorted in net-worth decile. The figure shows a pattern that is remarkably similar to the one predicted by the calibrated model and the one found in U.S. data (see Figure 5 in the

[3]We thank Olympia Bover of the Bank of Spain for pointing out this result to us and providing the data for Figure IA.7.

[4]In fact the questionnaire and fieldwork were contracted to NORC at the University of Chicago to benefit from the experience that NORC has acquired over the years when conducting the Survey of Consumer Finances in the US since 1993 on behalf of the Federal Reserve.

[5]A description in english of the methodology and data of the survey can be found in the Economic Bulletin of the Bank of Spain for January of 2005. See also Bover (2004). In addition, the characteristics of the survey led the European system of central banks to adopt it an extend it to the EU countries in the Household Finance and Consumption Surveys, allowing for comparisons of the Spanish surveys with European ones as well.

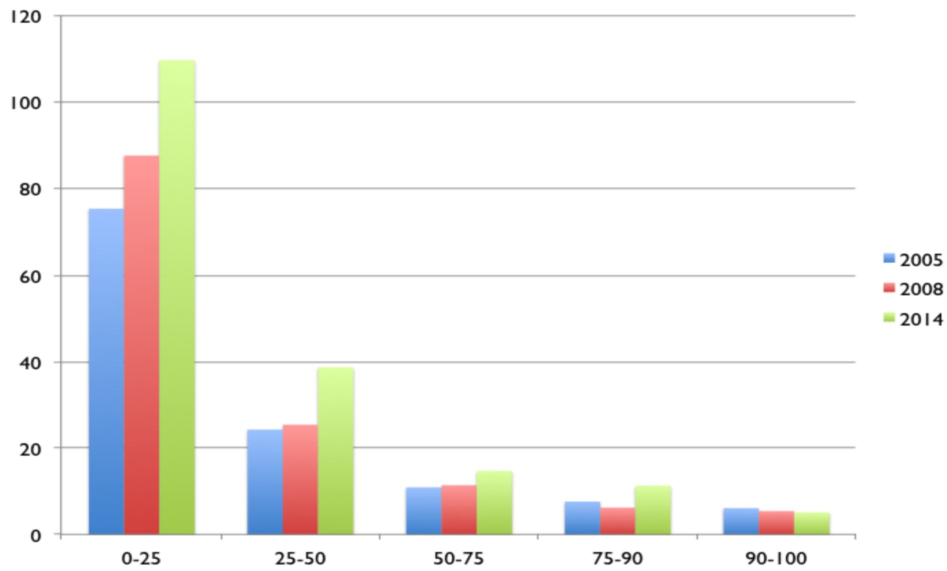


Figure IA.7: Household Leverage during the Spanish Crisis. This figure shows household leverage in Spain from the surveys of 2005 (boom), 2008 (slowdown), and 2014 (trough). Household leverage is the median ratio of debt to gross household wealth. The years correspond to three waves of the survey, which do not exactly match when the fieldwork took place. Data source: Encuesta Financiera de las Familias (EFF).

main text).[6] First, households with lower net worth are the most leveraged and leverage declines monotonically with net worth. Richer agents lever less. Second, Spanish households levered in good times, between 2005 and 2008, when the Spanish economy is booming (the peak of the economic cycle in Spain is 2006-2007 and the economy started slowing down in 2008, though still posted a positive growth rate for the year).

The Spanish crisis of 2012 and 2013 was deep and resulted in a severe drop in housing prices. Households though were not able to delever fast enough, either through debt repayment and default. The level of household debt peaked in the last quarter of 2008 and has been declining ever since.[7] Figure IA.7 shows that even in the presence of debt repayment (or personal default, which is much more onerous in Spain than in the US) households have not decreased their leverage. The reason of course, is the precipitous drop of housing prices. As a result even though debt repayment continues leverage has not dropped amongst a large swath of Spanish households.

[6]The second and third waves of the EFF have a full panel component; after that, and in order to main the representativeness of the sample, a refreshment sample was included. The plot shown thus differs slightly from the one reported with US data in that they are not exactly the same households across each of the years.

[7]Typically household debt for Spain is reported by reading loans extended by banks to consumers (mostly mortgages). that number peaked in the last quarter of 2008 at about EUR 800bn and it stands now at about EUR 650bn.

IA5. Discussion of the Model

Risk sharing and leverage are in our model two related but distinct concepts. Efficient risk sharing requires marginal utilities (scaled by the Pareto weights, ϕ_i; see equation (10)) to be equated across households (see equation (8)). How the competitive equilibrium implements the efficient allocation described in Proposition 1 depends on the specific financial market structure assumed and thus so do the leverage implications of our model. With this in mind, it is useful to consider how the portfolio allocations in Proposition 2 implement the efficient allocation described in Proposition 1 through a standard replication argument. Let W_{it} be the value of the contingent claim that at each point in time and state delivers as a dividend the consumption of agent i, C_{it}, associated with the efficient allocation (see equation (11)). We show in the Internet Appendix that the value of this contingent claim would be:

$$W_{it} = E_t \left[\int_t^\infty \frac{M_\tau}{M_t} C_{i\tau} d\tau \right] = \frac{\rho\gamma_i + (\rho(\omega_i - \gamma_i) + k\omega_i)\overline{I} I_t^{-1}}{\rho(\rho + k)} Y_t. \tag{IA.26}$$

Clearly a financial structure that features these contingent claims can equally implement the efficient allocation: Each agent would buy his corresponding contingent claim at date 0 and consume the dividends C_{it} throughout. Following Cox and Huang (1989) the stock investment and borrowing/lending decision in Proposition 4 simply replicates the cash-flows of this contingent claim

$$N_{it} P_t + (D_{it} - L_{it}) = W_{it}. \tag{IA.27}$$

where $(D_{it} - L_{it})$ is the net position in risk-free bonds. For this to be satisfied for every t (and pay C_{it} as dividend), it must be the case that the portfolio and the security have the same sensitivity to shocks dZ_t. Denoting by $\sigma_{Wi}(I_t)$ the volatility of $\log(W_{it})$, the portfolio allocation N_{it} and B_{it} must then satisfy

$$N_{it} = \frac{W_{it}\ \sigma_{Wi}(I_t)}{P_t\ \sigma_P(I_t)} \qquad \text{and} \qquad (D_{it} - L_{it}) = W_{it} - N_{it}P_t = W_{it}\left(1 - \frac{\sigma_{Wi}(I_t)}{\sigma_P(I_t)}\right). \tag{IA.28}$$

The net bond position, $(D_{it} - L_{it})$, depends on the ratio of volatilities $\frac{\sigma_{Wi}(I_t)}{\sigma_P(I_t)}$: If this ratio is greater than one, the agent is leveraging his investment in the stock market. The volatility of the contingent claim is

$$\sigma_{W_i}(I_t) = \sigma_Y(I_t)\left(1 + \frac{v\,(k + (\rho + k)(\omega_i/\gamma_i - 1))\,\overline{I}}{\rho I_t + (k + (\rho + k)(\omega_i/\gamma_i - 1))\,\overline{I}}\right). \tag{IA.29}$$

Comparing this expression with $\sigma_P(I_t)$ in (16), we see that $\sigma_{Wi}(I_t) > \sigma_P(I_t)$ if and only if $\omega_i > \gamma_i$. That is, RT households ($\omega_i > \gamma_i$) borrow to leverage their portfolio. Intuitively, from the optimal risk sharing rule (11), RT households have a high consumption share in good times, when I_t is low, and a low consumption share in bad times, when I_t is high. This particular consumption profile implies that the value of the contingent claim W_{it} is more sensitive to discount rate shocks than the stock price P_t. As a result the "replicating" portfolio requires some leverage to match such sensitivity.

Equation (IA.28) also highlights the reason why the aggregate debt-to-output ratio, which is equal to $\mathcal{D}_t/Y_t$, increases in good times. This is due to a "level effect": from (IA.29) and (16) the ratio of volatilities actually declines as I_t decreases. This is intuitive as the hypothetical contingent claim pays out more in good times and hence becomes less sensitive to discount rate shocks then. However, from (IA.26) the value of the hypothetical contingent claim W_{it} increases in good times because the discount rate declines and more than overcomes the decline in the ratio of volatilities. As a result, aggregate debt increases in good times.

While a procyclical aggregate debt-to-output ratio may seem intuitive, it is not normally implied by, for instance, standard CRRA models with differences in risk aversion. In such models, less risk averse households borrow from more risk averse agents, who want to hold riskless bonds rather than risky assets. As aggregate wealth becomes more concentrated in the hands of less risk-averse agents, the need of borrowing and lending declines, which in turn decreases aggregate debt.[8] Moreover, a decline in aggregate uncertainty – which normally occur in good times – actually decreases leverage in such models, as it reduces the risk-sharing motives of trade (see Veronesi (2018)). In our model, in contrast, the decrease in aggregate risk aversion in good times make households with high-risk bearing capacity even more willing to take on risk and hence increase their supply of risk-free assets to those who have a lower risk bearing capacity.

Finally, Figure IA.8 shows the conditional return distribution during booms (high S_t) and recessions (low S_t). During good times, the distribution is highly negatively skewed, implying that a crash is more likely than a boom. During bad times, the distribution is more uniform, due to the high stock return volatility.

[8]Longstaff and Wang (2012), Figure 5, shows the standard inverse-U shape relation between market leverage and the share of consumption of the least risk-averse agent s. Market leverage is minimized at the extremes for $s = 0$ and $s = 1$. While Longstaff and Wang do not report a similar plot for the debt-to-output ratio, it is straightforward verify that the same inverse-U shape holds also for the debt-to-output ratio.

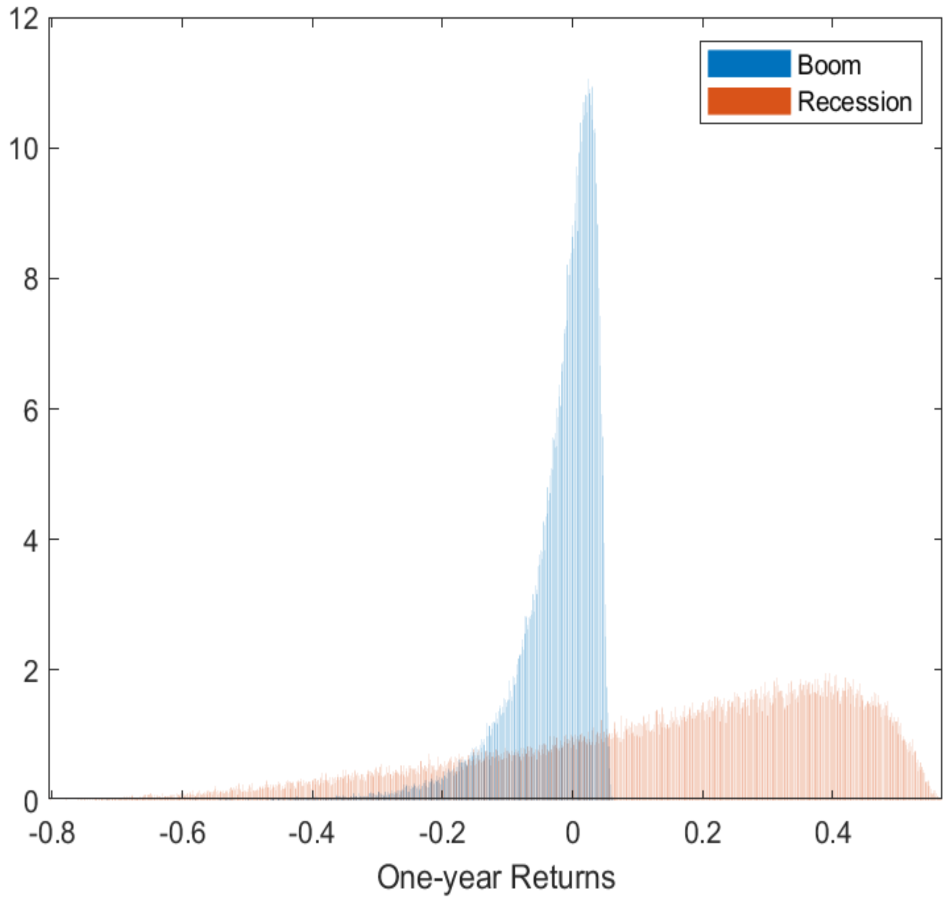


Figure IA.8: The Distribution of Stock Returns over the Business Cycle. This figures displays the conditional distribution of simulated stock returns in Booms and Recessions.

REFERENCES

Bank of Spain (2005) "Survey of Household Finances (EFF): description, methods and preliminary results," Economic Bulletin, January.

Bank of Spain (2017) "Survey of Household Finances (EFF) 2014: methods, results and changes since 2011," Bank of Spain, Analytical Articles.

Bover, Olympia (2004) "The Spanish Survey of Household Finances (EFF): description and methods of the 2002 wave," Bank of Spain, Occasional Paper no. 409.

Bover, Olympia (2011) "The Spanish Survey of Household Finances (EFF): description and methods of the 2008 wave," Bank of Spain, Occasional Paper no. 1103.

Cole, Rebel A. (2013) "What Do We Know about the Capital Structure of Privately Held US Firms? Evidence from the Surveys of Small Business Finance", *Financial Management*, Winter 2013, 777–813.

Graham, John R., Mark T. Leary, Michael R. Roberts, 2015, "A century of capital structure: The leveraging of corporate America," *Journal of Financial Economics*, 118, 658–683.