A New Class of Non-linear Term Structure Models

by Eraker, Wang and Wu

Discussion

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Main Contribution and Outline of Discussion

- Main contribution of the paper:
  - Proposes a new non-linear interest rate class of models
    - Based on Eraker and Wang non-linear model for volatility (on Journal of Econometrics)
  - Still preliminary version, so still developing main contribution
    - Better fit of term structure data?
    - Closed-form likelihood function for estimation?
    - Other?

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- Outline of discussion
  1. Review of new modeling device
  2. Review of other non-linear models
  3. Additional comments on the model

Model

- Short-term rate

$$r_t = f(x_t)$$

- State variable x_t follows some type of "easy process"

$$dx_t = \mu(x)dt + \sigma(x)dW_t$$

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- Issue: How do we solve for

$$B(t,T) = E^Q\left[e^{-\int_t^T f(x_u)du}\right] \ ?$$

- Change of numeraire: Define new numeraire $N = G(x)$ and let

$$P(x,t;T) = \frac{B(x;t,T)}{G(x)}$$

- Then we know

$$\mu(x)G'(x) + \frac{1}{2}\sigma^2(x)G''(x) = G(x)f(x)$$

Model

- Therefore, we can define a new measure N so that

$$dx_t = \left(\mu(x) + \sigma^2(x)\frac{G'(x)}{G(x)}\right)dt + \sigma(x)dW_t^N$$

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- Eraker, Wong, and Wu show that one can write this as

$$B(x,t;T) = \frac{1}{2\pi}G(x)\int_{-\infty}^{\infty} 1/\widehat{G}(u)\phi(t,u;x_t;T)du$$

where $\phi(.)$ is conditional characteristics function of x under new measure, and $1/\widehat{G}(u)$ is the Fourier transform of $1/G(x)$.

- $\Longrightarrow$ Judicious choices of $G(x)$, $\mu(x)$, $\sigma(x)$, $f(x)$ leads to closed form formulas.

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- $\Longrightarrow$ Judicious choices of $G(x)$, $\mu(x)$, $\sigma(x)$, $f(x)$ leads to closed form formulas.
- Example:

$$\mu(x) = k(\theta - x); \quad \sigma(x) = \sigma\sqrt{x}; \quad G(x) = x^{\alpha}e^{\beta x};$$

$$r_t = f(x_t) = \frac{a}{x} + bx + c$$

- Estimated non-linear model does better than CIR model

Other Non-Affine Models

- **Linear-quadratic models** (e.g. Ahn, Dittmar, Gallant 2002)

$$
\begin{aligned}
&\text{(factors)} \qquad & d\mathbf{X}_t &= \mathbf{K}\left(\theta - \mathbf{X}_t\right)dt + \mathbf{\Sigma} dW_t \\
&\text{(short rate)} \qquad & r_t &= \delta_0 + \delta_1' \mathbf{X}_t + \frac{1}{2}\mathbf{X}_t' \delta_{\mathbf{2}} \mathbf{X}_t
\end{aligned}
$$

  - Closed form formulas for bond prices

$$B(\mathbf{X}_t, t; T) = \exp\left(A(t,T) + \mathbf{B}(t,T)'\mathbf{X}_t + \frac{1}{2}\mathbf{X}_t' \mathbf{C}(t,T)\mathbf{X}_t\right)$$

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- **Linearity-generating model** (Gabaix (2007), Pancost (2015))

$$\begin{aligned} &\text{(factors)} & d\mathbf{X}_t &= \left[-\mu - \phi\mathbf{X}_t + \mathbf{X}_t'\mathbf{X}_t'\delta_1 + \boldsymbol{\Sigma}(\mathbf{X}_t)\sigma(\mathbf{X})\right] dt + \boldsymbol{\Sigma}(\mathbf{X} d\mathbf{W}_t \\ &\text{(SDF)} & \frac{dM_t}{M_t} &= -[\delta_0 + \delta_1'\mathbf{X}_t]dt - \sigma(\mathbf{X}_t) d\mathbf{W}_t \end{aligned}$$

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$$B(\mathbf{X}_t, t; T) = A(t,T) + \mathbf{B}(t,T)'\mathbf{X}_t$$

Other Non-Affine Models

- **Habit-based Term Structure Models** (Buraschi and Jiltsov (2007))

$$\begin{aligned} \text{(Inverse Surplus)} \qquad dY_t &= k(\overline{Y} - Y_t)dt - (Y_t - \lambda)\sigma_y dW_t \\ \text{(Money Shocks)} \qquad d\ell_{i,t} &= k_i(\overline{\theta}_i - \ell_{i,t})dt + \ell_{it}\sigma_i dW_{i,t} \end{aligned}$$

  - Closed form formula for bond prices

$$B(t,T) = e^{h(T-t)} \frac{\Sigma_{i=1}^{2} A_{i,0} + A_{i,1}Y_t + A_{i,2}\ell_i + A_{i,3}Y_t\ell_i}{\Sigma_{i=1}^{2} B_{i,0} + B_{i,1}Y_t + B_{i,2}\ell_i + B_{i,3}Y_t\ell_i}$$

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- **State-dependent, learning-based models** (Veronesi (2004))

$$\text{(beliefs)} \qquad d\pi_t = \mathbf{\Lambda}'\pi_t dt + \pi_t . * (\nu - \mathbf{1}_n \pi_t' \nu)\, d\mathbf{W}_t$$

  – Closed form formula for bond prices

$$B(t,T) = \frac{\pi_t'\, \mathbf{Q}(T-t)}{\pi_t'\, \mathbf{k}}$$

Zero-Lower Bound

- Need of non-linear models to deal with interest rates at zero lower bound
- **Shadow rate models** (Black (1995), Xia and Wu (2014))

$$
\begin{aligned}
\text{(factors)} \qquad & d\mathbf{X}_t = \mathbf{K}\left(\theta - \mathbf{X}_t\right) dt + \mathbf{\Sigma} dW_t \\
\text{(short shadow rate)} \qquad & s_t = \delta_0 + \delta_1' \mathbf{X}_t \\
\text{(short rate)} \qquad & r_t = \max(s_t, 0)
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- **VARG-Zero Models** (Monfort, Pegoraro, Renne, Rousellet 2014)

$$\begin{aligned} \text{(factors)} \qquad X_{j,t}|_{\mathbf{X}} &\sim \gamma_{\nu_j}(\delta_0 + \delta_1 \mathbf{X}_t, \mu_j) \\ \text{(transition)} \qquad \gamma_{\nu_j}(\lambda, \mu) &= \text{non-central Gamma distribution} \\ & \quad \nu_j = 0 \;\; j = 1, .., n; \quad \nu_j \geq 0 \;\; j = n+1, ..., M; \\ \text{(short rate)} \qquad r_t &= \sum_{j=1}^{n} \delta_j X_{j,t} \end{aligned}$$

  – Closed form formula for bond prices

$$B(\mathbf{X}_t, t; T) = \exp\left(A(t,T) + \mathbf{B}(t,T)'\mathbf{X}_t\right)$$

Positioning of the paper

- Why do we need a new class of non-linear models? What bad features of old models do we want to fix? What is the right benchmark? Just CIR model?
  - The new methodology is interesting, but as the paper shapes up, it would be great to figure out what new features of the data the new model is able to match
  - This is a class of models: Can we choose $f(x)$ so as to deal with the Zero Lower Bound and still get closed form solutions?

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- How about the market price of risk?
  - Non-linear (or non-affine) models claim they are better able to fit risk premia, and predict future returns.
  - How would you specify the market price of risk in this setting in order to get good results under the physical measure?
  - Is the flexibility afforded by the model able to generate any new insight?

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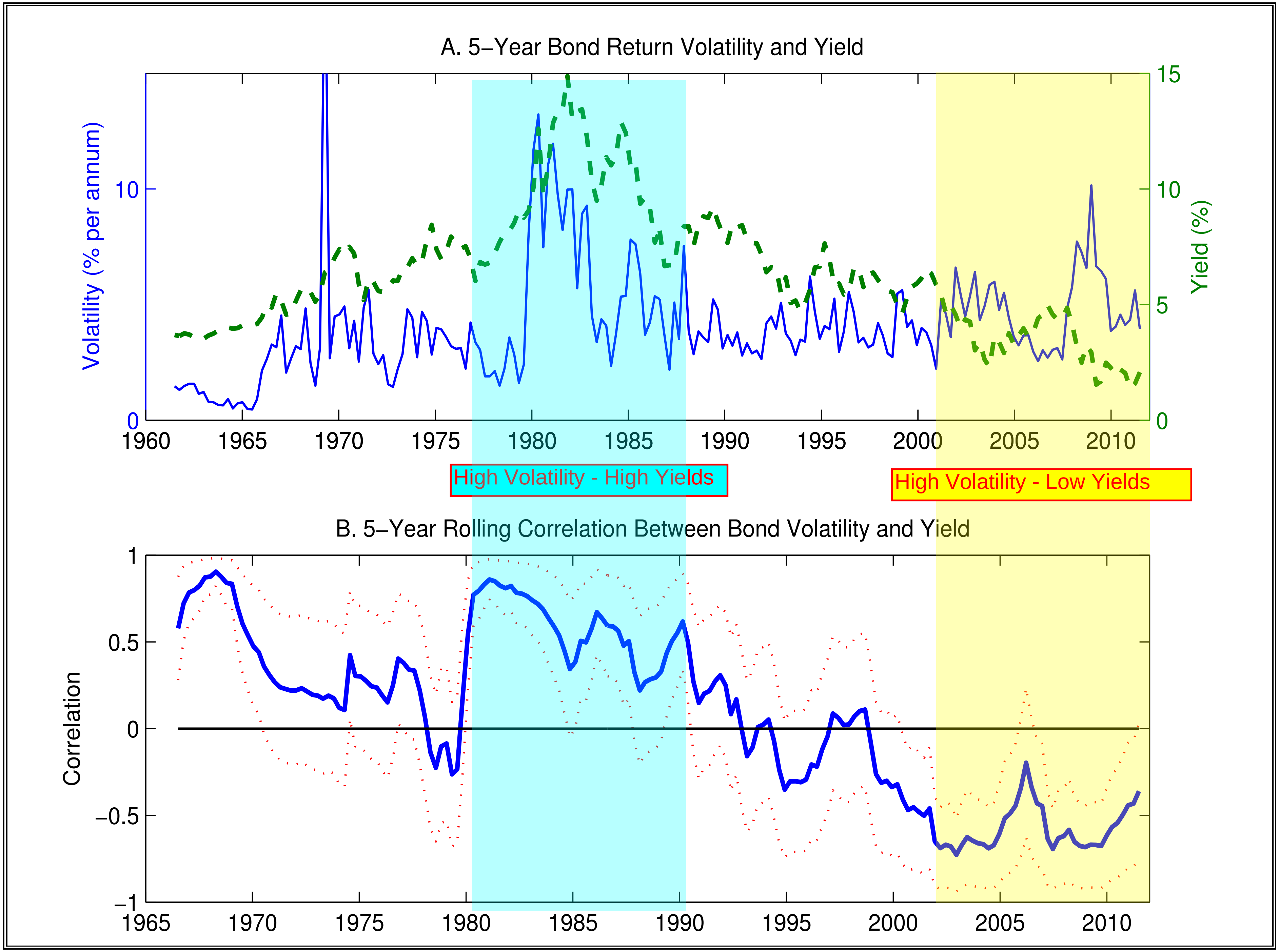
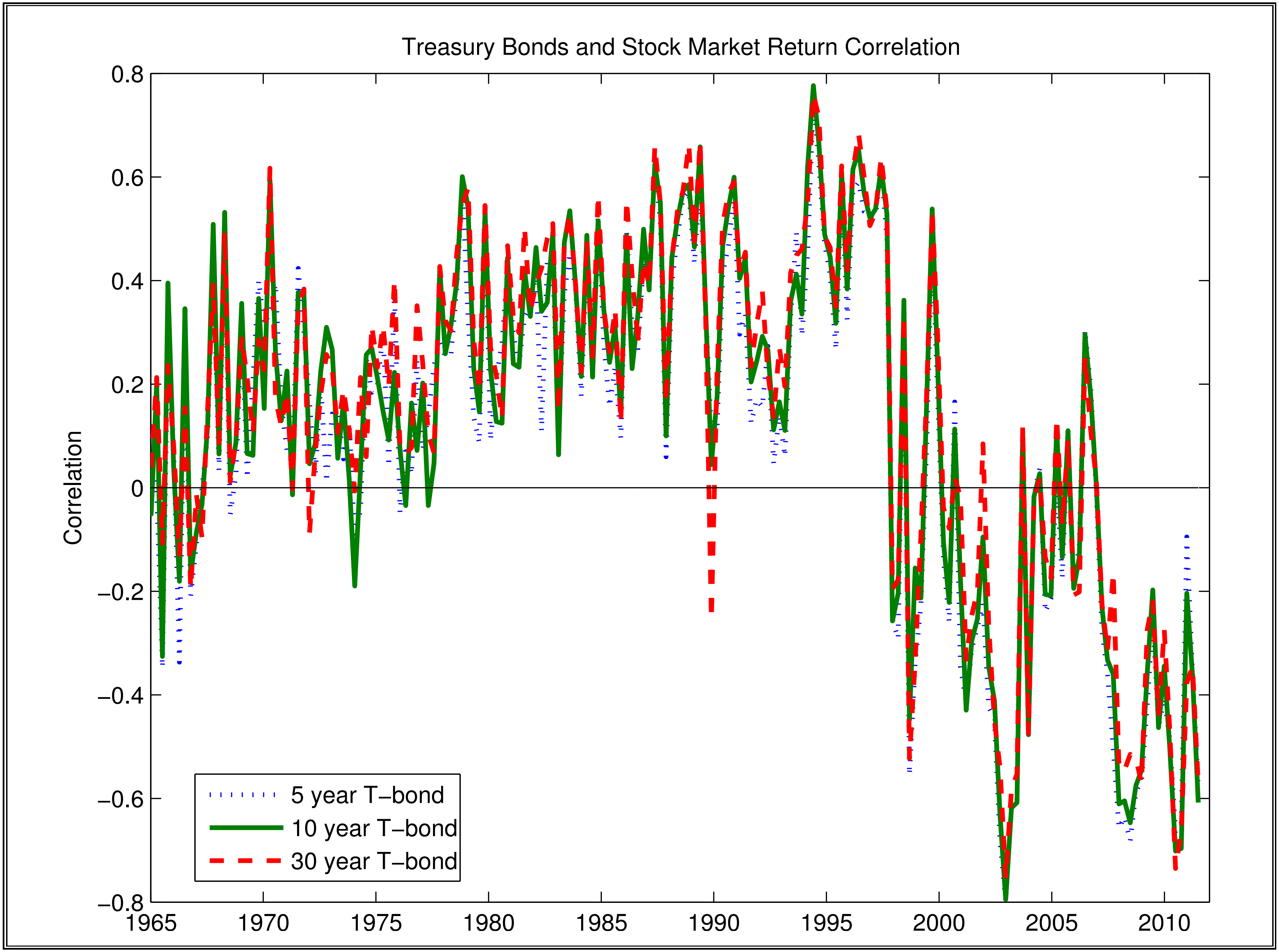
- Related, how about the issue of unspanned volatility?
  - Non-linear function $f(x)$ may generate a special relation between interest rate volatility and yields.
  - Can you use the flexibility of the model in order to ensure that volatility does not explain future returns, but it does generate time varying volatility?

Positioning of the paper

- Can the model generate the inversion in bond volatility and yields that we observed in the data?
  - 1980: High yields and high bond return volatility
  - 2010: Low yields and high bond return volatility
- Can the flexibility of the model generate time variation in the covariance between bonds and stocks?
  - 1980: Positive covariance
  - 2010: Negative covariance


A. 5−Year Bond Return Volatility and Yield
Volatility (% per annum)
Yield (%)
0
10
0
5
10
15
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
High Volatility - High Yields
High Volatility - Low Yields
B. 5−Year Rolling Correlation Between Bond Volatility and Yield
Correlation
1
0.5
0
−0.5
−1
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010


Treasury Bonds and Stock Market Return Correlation
Correlation
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
5 year T−bond
10 year T−bond
30 year T−bond

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- Can the flexibility of the model generate time variation in the covariance between bonds and stocks?
  - 1980: Positive covariance
  - 2010: Negative covariance
- How about the pricing of derivatives? Would the change in measure also help for pricing interest rate derivatives?
  - I am not sure the methodology can be easily applied to derivatives

$$V = G(x) E^N \left[\frac{\text{payoff}(x_T)}{G(x_T)} \right]$$

Overall....

- Overall: It is a promising approach, although it is too early to see through its full potential
  - I am a big fan of non-linear model, especially those that let risk prices change and switch sign over time
    - Such time variation seems needed to just fit the time series data in the past 30 years
- I look forward to receiving a more complete paper and see what the new methodology can really do.